# FGSI'19 Pannel session
## Symplectic Geometry & Localization in Mathematical Physics

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Montpellier

February 4, 2019

# Symplectic geometry

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## Official father is Hermann Weyl (1939)

> *The name "complex group" formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word "complex" in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective "symplectic".*

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Space of trajectories

$$\mathcal{T} = \{\big(q(t), p(t)\big)_{t\in[0,1]} | sol.\ of\ (1)\}$$

is Lagrangian in $R^{2n} \times R^{2n}$: $\mathcal{T}$ is the graph of the time 1 flow of (1).

# Localization

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This is just something about the $\frac{df}{\hbar}$-twisted de Rham cohomology.

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