Symplectic Diffeology

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Why Symplectic "Diffeology" ?

- "Symplectic Geometry" deals with symplectic manifolds and Lie groups actions, only.
- Last decades a lot of examples of "symplectic" structures has been built on spaces which are not manifolds.
- We may want to consider actions of groups that are not Lie groups, even acting on manifolds.
- Symplectic reduction introduces quotient spaces that are not manifolds but carry "parasymplectic" structures.

Example 1 The Moment of Imprimitivity

Action of $C^\infty(M, \mathbf{R})$ on T^*M

- M is a manifold, $q \in M$.
- T^*M is the cotangent space, $(q, p) \in T^*M$.
- $f \in C^\infty(M, \mathbf{R})$.

$$\underline{f}(q, p) = (q, p + df(q)).$$

Question : Moment Map ?

Example 2 Intersection Form on a Surface I

Action of $C^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$

- Σ is a oriented surface.
- $\Omega^1(\Sigma)$ is the space of 1-forms on Σ, $\alpha, \beta \in \Omega^1(\Sigma)$.

$$\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta.$$

- Group $C^\infty(\Sigma, \mathbf{R})$, action:

$$f \in C^\infty(\Sigma, \mathbf{R}), \alpha \in \Omega^1(\Sigma), \quad \underline{f}(\alpha) = \alpha + df.$$

Question : Is that a symplectic space ? In what sense ?
Question : Moment Map ?

Example 3 Intersection Form on a Surface II

Action of $\text{Diff}(\Sigma)$ on $\Omega^1(\Sigma)$

- Σ is a oriented surface.
- $\Omega^1(\Sigma)$ is the space of 1-forms on Σ, $\alpha, \beta \in \Omega^1(\Sigma)$.

$$\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta.$$

- Group $\text{Diff}^+(\Sigma)$, action:

$$\phi \in \text{Diff}^+(\Sigma), \alpha \in \Omega^1(\Sigma), \quad \underline{\phi}(\alpha) = \phi_*(\alpha).$$

Question : Moment Map ?

Example 4 Intersection Form on a Surface III

Action of $\Omega^1(\Sigma)$ on itself

- Σ is a oriented surface.
- $\Omega^1(\Sigma)$ is the space of 1-forms on Σ, $\alpha, \beta \in \Omega^1(\Sigma)$.

$$\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta.$$

- Group $\Omega^1(\Sigma)$, action:

$$\beta \in \Omega^1(\Sigma), \alpha \in \Omega^1(\Sigma), \quad \underline{\beta}(\alpha) = \alpha + \beta.$$

Question : Moment Map ?

Example 5 Symplectic Torus

Moment Map for the Torus

- T^2 is the 2-torus $\mathbf{R}^2/\mathbf{Z}^2$.
- $\omega \in \Omega^2(\mathsf{T}^2)$ is the pushforward of $dx \wedge dy$.
- T^2 is a group and act on itself preserving ω, action:

$$\tau, z \in \mathsf{T}^2 \quad \underline{\tau}(z) = z + \tau.$$

Question : Moment Map for non-Hamiltonian action ?
Question : Generalization to $\mathbf{R}^n/\mathsf{K}$, K subgroup ?

Example 6 Orbifolds

2-Forms on Orbifolds

- An orbifold is locally a quotient $\mathbf{R}^n/\Gamma$, $\Gamma \mathrm{GL}(n)$ finite.
- The *Cone Orbifold* $\mathcal{Q}_m = \mathbf{C}/\mathrm{U}_m$, U_m m-roots of unity.
- The *Corner Orbifold* $\mathcal{Q} = [\mathbf{R}/\pm 1]^2$.

Question : $\Omega^2(\mathrm{Q}_m)$? $\Omega^2(\mathrm{Q})$?
Question : Group actions, Moment maps ?

Example 7 Infinite Projective Space

The infinite Sphere and its Quotient

- $\mathcal{H} = \{Z = (Z_i)_{i=1}^{\infty} \mid \sum_{i=1}^{\infty} \bar{Z}_i \cdot Z_i < \infty\}$, Hermitian product.
- $S^{\infty} = \{Z \in \mathcal{H} \mid \bar{Z} \cdot Z = 1\}$.
- $\alpha = \frac{1}{2i}[\bar{Z} \cdot dZ - d\bar{Z} \cdot Z]$
- $\mathbf{C}P^{\infty} = S^{\infty}/S^1$

Question : Reduction of $d\alpha \upharpoonright S^{\infty}$ on $\mathbf{C}P^{\infty}$?
Question : Moment map of $U(\mathcal{H})$? And of S^1 ?
Question : Is $\mathbf{C}P^{\infty}$ symplectic ?

Example 8 Virasoro et al.

Immersing S^1 in $\mathbf{R}^2$

- $\mathrm{Imm}(\mathsf{S}^1, \mathbf{R}^2) = \{x \in \mathsf{C}^\infty(\mathsf{S}^1, \mathbf{R}^2) \mid \dot{x}(t) \neq 0\}$.
- $\alpha \in \Omega^1(\mathrm{Imm}(\mathsf{S}^1, \mathbf{R}^2))$

$$\alpha(\delta x) = \int_0^{2\pi} \frac{1}{\|\dot{x}(t)\|^2} \langle \ddot{x}(t) | \delta\dot{x}(t) \rangle \, dt.$$

- Group $\mathrm{Diff}(\mathsf{S}^1)$ acts on $\mathrm{Imm}(\mathsf{S}^1, \mathbf{R}^2)$: $\varphi \in \mathrm{Diff}(\mathsf{S}^1)$, $x \in \mathrm{Imm}(\mathsf{S}^1, \mathbf{R}^2)$, $\underline{\varphi}(x) = x \circ \varphi^{-1}$.
- α not invariant but $d\alpha$ is: $\underline{\varphi}^*(d\alpha) = d\alpha$.

Question : Moment map of $\mathrm{Diff}(\mathsf{S}^1)$? Equivariance ?
Question : Souriau's cocycle θ ?
Question : "Symplectic" reduction ?

Example 9 Reducing $C^\infty(S^1, \mathbf{C})$.

Reducing Complex Periodic Functions by a Real flow

- $C^\infty(S^1, \mathbf{C}) \simeq \{f \in C^\infty(\mathbf{R}, \mathbf{C}) \mid f(x+1) = f(x)\}$
- Equivalent to $\mathcal{E} = \{(f_n)_{n\in Z} \mid f_n \downarrow 0\}$, rapidly decreasing.
- $\epsilon \in \Omega^1(C^\infty(S^1, \mathbf{C}))$

$$\epsilon(\delta f) = \int_0^1 \bar{f}(x)\delta f(x)\, dx, \quad d\epsilon = \frac{1}{\pi}\int_0^1 \hat{x}^*(\mathrm{surf})\, dx$$

- Group $\mathbf{R}$ acts on $\mathcal{E}$: $\underline{t}\big((f_n)_{n\in Z}\big) = \big(e^{2i\pi\alpha_n t} f_n\big)_{n\in Z}$, with α_n independant on $\mathbf{Q}$.

Question : Orbits of $\mathbf{R}$?
Question : Moment map h of $\mathbf{R}$?
Question : Reduction of $S^\infty_\alpha = h^{-1}(1)$?

Category Diffeology I

A *diffeology* on a set X is given by a choice $\mathcal{D}$ of *Euclidean Parametrizations* called *Plots*, satisfying 3 axioms.

- The plots cover X.
- To be a plot is local.
- The composite of a plot by a smooth parametrization of its domain is a plot.

A set X equipped with a diffeology $\mathcal{D}$ is called a *diffeological space*.

- A map $f\colon \mathrm{X} \to \mathrm{X}'$, where X and X' are two diffeological spaces, is said to be *smooth* if the composite of f with a plot of X is a plot of X'.

Symplectic Framework II

Category Diffeology II

Diffeological spaces together with smooth maps define the category {Diffeology}. Isomorphisms are called *diffeomorphisms*. This category is stable for the set theoretic operations.

- Sums: $\coprod_i \mathrm{X}_i$.
- Products: $\prod_i \mathrm{X}_i$.
- Quotients: $\mathcal{Q} = \mathrm{X}/\!\sim$.
- Subsets: $\mathrm{A} \subset \mathrm{X}$.

It is a complete and co-complete category.
The set $\mathrm{C}^\infty(\mathrm{X}, \mathrm{X}')$ has a natural *Functional Diffeology* which makes {Diffeology} Cartesian closed.

Differential Forms

Differential Forms on Diffeological Spaces

- A *differential k-form* α on a diffeological space X is a mapping that associates with each plot P in X, a smooth form

$$\alpha(\mathrm{P}) \in \Omega^k(\mathrm{dom}(\mathrm{P})),$$

such that, for all smooth parametrization F in dom(P)

$$\alpha(\mathrm{P} \circ \mathrm{F}) = \mathrm{F}^*\big(\alpha(\mathrm{P})\big).$$

- Let $f \colon \mathrm{X} \to \mathrm{X}'$ be a smooth map, and $\alpha' \in \Omega^k(\mathrm{X}')$. Then,

$$\Omega^k(\mathrm{X}) \ni f^*(\alpha') \colon \mathrm{P} \mapsto f^*(\alpha')(\mathrm{P}) = \alpha'(f \circ \mathrm{P}).$$

Diffeological Groups

Diffeological Groups And Momenta

- A *Diffeological Group* is a diffeological space together with a group law such that the product and the inverse are smooth.
- Every group of diffeomorphisms $\mathrm{Diff}(\mathrm{X})$ is a diffeological group, equipped with the functional diffeology.
- A *momentum* of a diffeological group G is a left-invariant 1-form. We introduce the vector *Space of Momenta*:

$$\mathcal{G}^* = \{\alpha \in \Omega^1(\mathrm{G}) \mid \forall g \in \mathrm{G}, \mathrm{L}(g)^*(\alpha) = \alpha\}.$$

  With $\mathrm{L}(g) \colon g' \mapsto gg'$.

➠ Note that the space of momenta $\mathcal{G}^*$ is not defined by duality with a presumed Lie algebra.

Moment Map I

Parasymplectic Forms and Moment Map, the **Simplest Case**

- Let us call *parasymplectic form* on a diffeological space X, any closed 2-form: $\omega \in \Omega^2(\mathrm{X})$ and $d\omega = 0$.
- Let G be a diffeological group acting smoothly on X and preserving ω: $\underline{g}^*(\omega) = \omega$, for all $g \in \mathrm{G}$.

➠ Now assume that $\omega = d\alpha$ and $\underline{g}^*\alpha = \alpha$. Let $\hat{x} \colon g \mapsto \underline{g}(x)$ be the *orbit map*.

- The map $\mu \colon x \mapsto \hat{x}^*(\alpha)$, defined on X is smooth and takes its values in $\mathcal{G}^*$. It is the Moment Map of ω.

➠ Actually it is a moment map. The moment map associated with the primitive α.

Moment Map II

Parasymplectic Forms and Moment Map, the **General Case**

- There exists a *Chain-Homotopy Operator*

$$\mathcal{K} \colon \Omega^k(\mathsf{X}) \to \Omega^{k-1}(\mathsf{Paths}(\mathsf{X}))$$

  such that

$$d \circ \mathcal{K} + \mathcal{K} \circ d = \hat{1}^* - \hat{0}^*,$$

  where $\mathsf{Paths}(\mathsf{X}) = \mathsf{C}^\infty(\mathbf{R}, \mathsf{X})$, $\hat{t}(\gamma) = \gamma(t)$, for $t \in \mathbf{R}$.
- Then,

$$\bar{\omega} = d\lambda, \text{ with } \lambda = \mathcal{K}\omega \text{ and } \bar{\omega} = \hat{1}^*(\omega) - \hat{0}^*(\omega).$$

- If G preserves ω, then G preserve $\lambda = \mathcal{K}\omega$.
- Hence, we are brought back to the simplest case $\underline{g}^*(\lambda) = \lambda$.

Moment Map III

The Paths-Moment Map

- The *Paths-Moment Map* is defined by:

$$\Psi\colon \mathrm{Paths}(X) \to \mathcal{G}^* \quad \text{with} \quad \Psi(\gamma) = \hat{\gamma}^*(\mathcal{K}\omega).$$

- It is G-equivariant for the coadjoint action. Let $g, k \in G$ and $\alpha \in \mathcal{G}^*$.

$$\mathrm{ad}(g)(k) = gkg^{-1}, \quad \mathrm{Ad}_*(g)(\alpha) = \mathrm{ad}(g)_*(\alpha).$$

- It is additive:

$$\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma'),$$

for all γ and γ' juxtaposable.

Moment Map IV

The Two-Points-Moment Map

- The *Two-Points-Moment Map* is defined[2]: by:

$$\psi\colon \mathrm{X} \times \mathrm{X} \to \mathcal{G}^*/\Gamma, \quad \text{with} \quad \psi(x, x') = \Psi(\gamma),$$

  with $x = \gamma(0)$ and $x' = \gamma(1)$.
- $\Gamma \subset \mathcal{G}^*$ is made of Ad_*-invariant momenta. It it the *Holonomy* of the action of G on (X, ω), the obstruction of the action of G for being *Hamiltonian*.

$$\Gamma = \{\Psi(\ell) \mid \ell \in \mathrm{Loops}(\mathrm{X})\}.$$

- ψ is still G-equivariant and a cocycle:

$$\psi(x, x') + \psi(x', x'') = \psi(x, x'').$$

[2] X is assumed connected.

Moment Map V

The One-Point-Moment Map

- A *One-Point-Moment Map* is a solution μ of

$$\psi(x, x') = \mu(x') - \mu(x) \quad \text{with} \quad \mu\colon X \to \mathcal{G}^*/\Gamma.$$

That is:

$$\mu(x) = \psi(x_0, x) + c, \text{ where } x_0 \in X \text{ and } c \in \mathcal{G}^*/\Gamma.$$

- The moment map μ is θ-affine Ad_*-equivariant:

$$\begin{aligned} \mu(\underline{g}(x)) &= \mathrm{Ad}_*(\mu(x)) + \theta(g), \text{ with} \\ \theta(g) &= \psi(x_0, \underline{g}(x_0)) - \Delta(c)(g), \end{aligned}$$

$\theta \in H^1(G, \mathcal{G}^*/\Gamma)$, and $\Delta(c)(g)$ is the coboundary $\mathrm{Ad}_*(g)(c) - c$.

Presymplectic Spaces

From Parasymplectic to Presymplectic – I

- A presymplectic manifold (M, ω) satisfies the *passive* version of the Darboux theorem : the existence of Darboux charts where ω takes the form

$$[\omega] = \begin{pmatrix} 0_k & \mathbf{1}_k & 0_{k,r} \\ -\mathbf{1}_k & 0_k & 0_{k,r} \\ 0_{r,k} & 0_{r,k} & 0_r \end{pmatrix}.$$

- Equivalently, we have the *active* version of the Darboux theorem :

$\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M}, \omega)$ is transitive on M.

From Parasymplectic to Presymplectic – II

In diffeology we do not like very much contravariant objects, and the kernel of a form is contravariant. That's why we define:

Definition

A parasymplectic diffeological space (X, ω) will be said *presymplectic* if its pseudo-group of local automorphisms $\mathrm{Diff}_{\mathrm{loc}}(X, \omega)$ is transitive. We shall call that condition the *Darboux property*.

More precisely, we ask that X is locally homogeneous under $\mathrm{Diff}_{\mathrm{loc}}(M, \omega)$.

Symplectic Spaces

From Presymplectic to Symplectic

Consider a presymplectic manifold (M, ω).

Theorem

The presymplectic form ω is symplectic if and only if the universal moment map μ_ω of the group Diff(M, ω) is injective.

The two conditions are necessary as show the example on $\mathbf{R}^2$,

$$\omega = (x^2 + y^2) dx \wedge dy$$

The universal moment is injective but $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{M}, \omega)$ is not transitive, $(0, 0)$ is fixed.

Symplectic Spaces

Symplectic Diffeological Spaces

Consider a parasymplectic diffeological space (X, ω).

Definition

We shall say that the form ω is symplectic if and only if the two conditions are satisfied:

- The pseudo-group $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}, \omega)$ is transitive.
- The universal moment map μ_ω of the group $\mathrm{Diff}(\mathrm{X}, \omega)$ is injective or at most with diffeologically discrete preimages.

Note that according to this definition, because its origin is fixed, the cone orbifold $\mathcal{Q}_m$ is not symplectic, contrary to what is admitted usually.

The Moment of Imprimitivity

Action of $C^\infty(M, \mathbf{R})$ on T^*M

- The group $C^\infty(M, \mathbf{R})$ acts on T^*M by $\underline{f}(q, p) = (q, p + df(q))$, $f \in C^\infty(M, \mathbf{R})$. It preserves the 2-form $\omega = dp \wedge dq$.
- The moment map is

$$\mu\colon (q, p) \mapsto d[f \mapsto f(q)].$$

  Note: $[f \mapsto f(q)] \in C^\infty(C^\infty(M, \mathbf{R}), \mathbf{R})$ is not invariant, but its differential is an invariant 1-form on $C^\infty(M, \mathbf{R})$.
- Also $\mu(q, p) = d\delta_q$, where δ_q is the Dirac function. The moment map is the differential of a distribution.

Intersection Form on a Surface I

Action of $C^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$

- The group $C^\infty(\Sigma, \mathbf{R})$ acts on $\Omega^1(\Sigma)$, preserving the 2-form $\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta$.
- For all $f \in C^\infty(\Sigma, \mathbf{R})$, $\alpha \in \Omega^1(\Sigma)$, $\underline{f}(\alpha) = \alpha + df$.
- The moment map is:

$$\mu\colon \alpha \mapsto d\left[f \mapsto \int_\Sigma f \, d\alpha\right].$$

- Again, the moment map is the differential of a distribution: $f \mapsto \int_\Sigma f \, d\alpha$.

Intersection Form on a Surface III

Action of $\Omega^1(\Sigma)$ on itself

- The group $\Omega^1(\Sigma)$ acts additively on itself, preserving the 2-form $\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta$.
- The moment map is:

$$\mu \colon \alpha \mapsto d\left[\beta \mapsto \int_\Sigma \alpha \wedge \beta\right].$$

- Here again, the moment map is the differential of a distribution.
- The space $\Omega^1(\Sigma)$ is symplectic in the sense above.
  1. The group of automorphisms is transitive.
  2. The moment map μ is injective.

Example 8 Virasoro et al. I

Immersing S^1 in $\mathbf{R}^2$

- We deal with $\omega = d\alpha$, with
$$\alpha(\delta x) = \int_0^{2\pi} \frac{1}{\|\dot{x}(t)\|^2} \langle \ddot{x}(t) | \delta\dot{x}(t) \rangle \, dt, \;\; x \in \mathrm{Imm}(S^1, \mathbf{R}^2).$$
The group $\mathrm{Diff}^+(S^1)$ acts on $\mathrm{Imm}(S^1, \mathbf{R}^2)$ by $\underline{\varphi}(x) = x \circ \varphi^{-1}$.
- On the connected component of the standard immersion $t \mapsto (\cos(t), \sin(t))$, the moment map is, up to a constant:
$$\mu(x)(P)_r(\delta r) = \int_0^{2\pi} \left\{ \frac{\|x''(u)\|^2}{\|x'(u)\|^2} - \frac{d^2}{du^2} \log \|x'(u)\|^2 \right\} \delta u \; du.$$
$P \colon r \mapsto \varphi$ is a n-plot of $\mathrm{Diff}_+(S^1)$, $r \in \mathrm{dom}(P)$, $\delta r \in \mathbf{R}^n$, $u = \varphi^{-1}(t)$, where t is the parameter of $x \in \mathrm{Imm}(S^1, \mathbf{R}^2)$, and $\delta u = D(r \mapsto u)(r)(\delta r)$.

Example 8 Virasoro et al. II

Immersing S^1 in $\mathbf{R}^2$

- The affine cocycle of the $\mathrm{Diff}_+(S^1)$ action on $\mathrm{Imm}(S^1, \mathbf{R}^2)$ are cohomologous to θ defined by,

$$\theta(g)(P)_r(\delta r) = \int_0^{2\pi} \frac{3\gamma''(u)^2 - 2\gamma'''(u)\gamma'(u)}{\gamma'(u)^2}\ \delta u\ du,$$

  where $g \in \mathrm{Diff}^+(S^1)$ and $\gamma = g^{-1}$. We recognize the Schwarzian derivative in the integrand of the right hand side.
- The cocycle θ of this integral construction of the moment map in diffeology, extends Souriau's cocycles of manifold cases.

For Further Reading

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