Localization in geometry and physics

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Gaussian integral

$$Z = \int_{-\infty}^{\infty} d\phi e^{-\frac{D}{2}\phi^2} = \sqrt{2\pi D^{-1}}$$

plays a remarkable role in information theory, probability, mathematics and physics

In fact, most of modern quantum field theory is built around Feynman path integral formulation:

$$Z[\theta] = \int D\Phi e^{-S[\phi,\theta]}$$

where we integrate of the space of *fields Φ;* while θ are parameters.

The integral is typically infinite-dimensional, which does not stop practitioners of QFT to get sensible results in agreement with experiment.

For example, current measured value of electron g-2 factor is

2.0023193043617(15)

and QFT (4-loop computation) based on independent measurement of 1/α = 137.035 998 78 (91) are in the excellent agreement: 10^(-8) precision

How do we think about path integrals like this ?

$$Z[\theta, \hbar] = \int D\Phi e^{-\frac{1}{\hbar}S[\phi,\theta]}$$

Here the factor called hbar is explicitly displayed.

A typical approach is to compute the asymptotic expansion of Z[θ,h] in the limit

$$\hbar \to 0$$

Assuming that S[ϕ,θ] is bounded from below and analytic in ϕ, consider the extremal point

$$\phi_* : \qquad \frac{\partial S}{\partial \phi} = 0, \qquad \frac{\partial^2 S}{\partial \phi^2} > 0$$

$$S = S_0[\phi_*] + \frac{1}{2}\frac{\partial^2 S}{\partial \phi^i \partial \phi^j}\phi^i \phi^j + \dots$$

where the dots denote the higher order term typically called interaction

$$S = S_0[\phi_*] + \frac{1}{2}\frac{\partial^2 S}{\partial\phi^i\partial\phi^j}\phi^i\phi^j + S_{int}[\phi]$$

The partition function reduces to sum of terms, where each term is the expectation value with respect to normal distribution

$$Z[\theta,\hbar] = e^{-\frac{1}{\hbar}S_0[\phi_*]}\sum_{k=0}^{\infty}\int d\phi\, e^{-\frac{1}{2\hbar}(\phi,D\phi)}\left(-\hbar^{-1}V[\phi]\right)^k$$

where $$D_{ij} = \frac{\partial^2 S}{\partial\phi^i\partial\phi^j}$$

each term in the expansion

$$Z[\theta, \hbar] = e^{-\frac{1}{\hbar} S_0[\phi_*]} \sum_{k=0}^{\infty} \int d\phi \, e^{-\frac{1}{2\hbar}(\phi, D\phi)} \left(-\hbar^{-1} V[\phi]\right)^k$$

is pictured by QFT practitioners as a Feynman diagram

Suppose that V[ϕ] contains a term like

$$V[\phi] = V_{ijk} \phi^i \phi^j \phi^k$$

and we compute the term

$$\langle V_{i_1 i_2 i_3} \phi^{i_1} \phi^{i_2} \phi^{i_3} V_{i_4 i_5 i_6} \phi^{i_4} \phi^{i_5} \phi^{i_6} \rangle$$

where

$$\langle O \rangle = \int d\phi \, e^{-\frac{1}{2\hbar}(\phi, D\phi)} O(\phi)$$

we need a basic variation of gaussian integral to compute

$$\int d\phi\, e^{-\frac{1}{2\hbar}(\phi, D\phi)} \phi^{i_1} \phi^{i_2} \ldots \phi^{i_n}$$

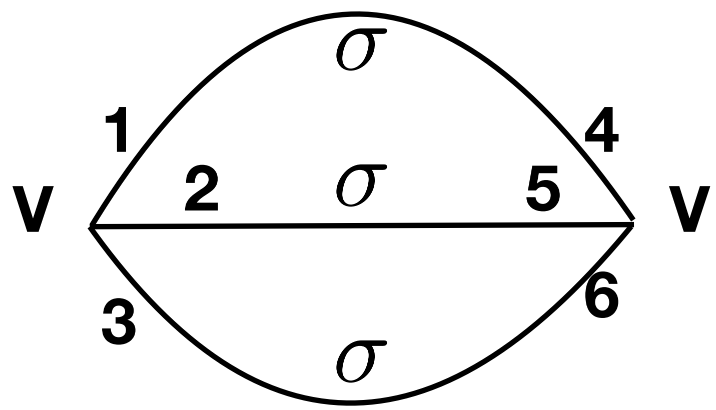
which is nonzero only if *n* is even, and then is given by

$$\frac{\int d\phi\, e^{-\frac{1}{2\hbar}(\phi, D\phi)} \phi^{i_1} \phi^{i_2} \ldots \phi^{i_n}}{\int d\phi\, e^{-\frac{1}{2\hbar}(\phi, D\phi)}} =$$

$$= \sum_{\substack{\text{(n-1)!! ways } \sigma \text{ of arranging } 1 \ldots n \text{ indices} \\ \text{into pairs } (\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_n))}} D^{-1}_{\sigma(i_1),\sigma(i_2)} \ldots D^{-1}_{\sigma(i_{n-1},i_n)}$$

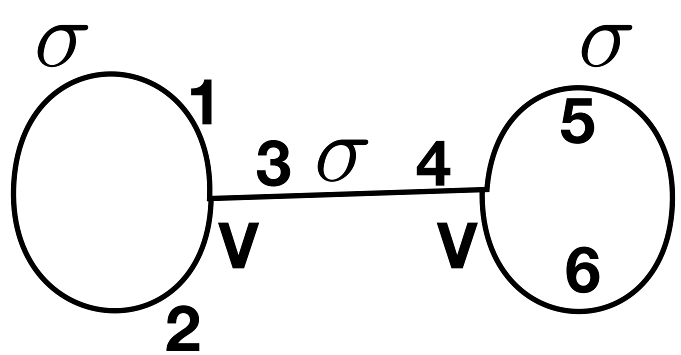
for example

$$\langle V_{i_1 i_2 i_3}\phi^{i_1}\phi^{i_2}\phi^{i_3}V_{i_4 i_5 i_6}\phi^{i_4}\phi^{i_5}\phi^{i_6}\rangle =$$



$$= \quad V_{i_1 i_2 i_3}V_{i_4 i_5 i_6}\sigma^{i_1 i_4}\sigma^{i_2 i_5}\sigma^{i_3 i_6}$$

+ permutations… (in total 6 graphs)



$$= V_{i_1 i_2 i_3}V_{i_4 i_5 i_6}\sigma^{i_1 i_2}\sigma^{i_3 i_4}\sigma^{i_5 i_6}$$

+ other elements of 3x3 set (in total 9 graphs)

where $\sigma = D^{-1}$

We have just seen how to obtain the asymptotic expansion of

$$Z[\theta, \hbar] = \int D\Phi e^{-\frac{1}{\hbar}S[\phi,\theta]}$$

where the intermediate steps require Gaussian integration.

Why Gaussian (normal) distribution is so omni-present in physics, mathematics and information theory?

The standard answer is that Gaussian distribution comes as a distribution of a sum of large number of whatever distributed variables as long as

- the variation of each variable is finite
- the variables are distributed independently

This is a content of the famous central limit theorem which was published by Laplace in 1812. There is a twist in a history of this theorem that we'll touch shortly.

In fact, as we shall see in the rest of the lecture, Gaussian functionals play play instrumental role in the differential geometry, symplectic geometry, enumerative geometry, algebraic topology, index theory, etc.

Moreover, in multiple cases, whenever something is exactly integrable, it turns out that there was a hidden Gaussian somewhere in the problem.

Why Gaussian is everywhere in physics?

Is it an accident?

What is relation to information theory?

information, complexity,
combinatorics
discrete
algebra

$\longleftrightarrow$

probability, dynamics,
statistical/quantum physics
continuous
geometry

The relation between discrete and continuous, between formula and shape, between algebra and geometry, was always in the heart of mathematics

One of the key discoveries was by de Moivre in 1733:

The the number of ways to choose *k* out of n elements C(n,k)=n!/((n-k)!k!) is approximated by normal (Gaussian) distribution in the limit of large (k,n).

$$\binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left(-\frac{(k-np)^2}{2np(1-p)}\right)$$

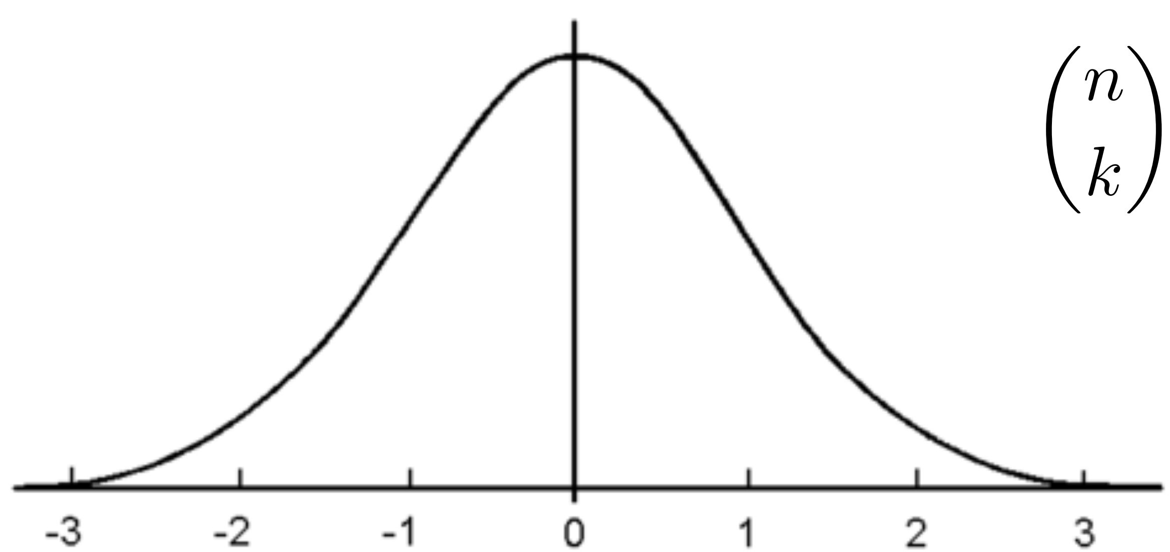
The left hand side of de Moivre formula involves intuitive combinatorial integral object: binomial coefficients C(n,k)

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

simple recurrent relation:

$$C(n,k) = C(n-1,k-1) + C(n-1,k)$$

The right hand side contains transcendental symbols: ‘e’, the base of natural logarithms and square root of ‘pi’, the ratio of circumference of circle to the diameter: magic?



$$\binom{n}{k} \simeq 2^n \frac{1}{\sqrt{\frac{1}{2}\pi n}} \exp\left(-\frac{(k-n/2)^2}{n/2}\right)$$

large *n*

small (k-n/2)/n

The consequences of the idea of de Moivre to look on the asymptotic limit are not yet exhausted ….

The diffusion, Brownian motion, entropy, stock market model, Feynman's path integrals, heat kernels grow from de Moivre observation

Gaussian distribution could be called de Moivre distribution,
- de Moivre's paper is in 1733 (published in "Doctrine of Chances" 1738)
- Gauss's paper on the maximal likelihood and linear regression by the method of least squares is in 1809

Perhaps, we've got so used to the formula found by de Moivre, that we sometimes forget how beautiful it is, especially in the time of its discovery

We found that Gaussian came from C(n,k). Where C(n,k) came from?

Now let us track the C(n,k) distribution to the historical roots !

The binomial coefficients C(n,k) have primarily information/combinatorial content: they count the number of sequences of length *n* composed on alphabet of two letters, say ‘L’ (light) and ‘G’ (heavy) which have exactly *k* letters ‘L’.

For example C(5, 3) = 10

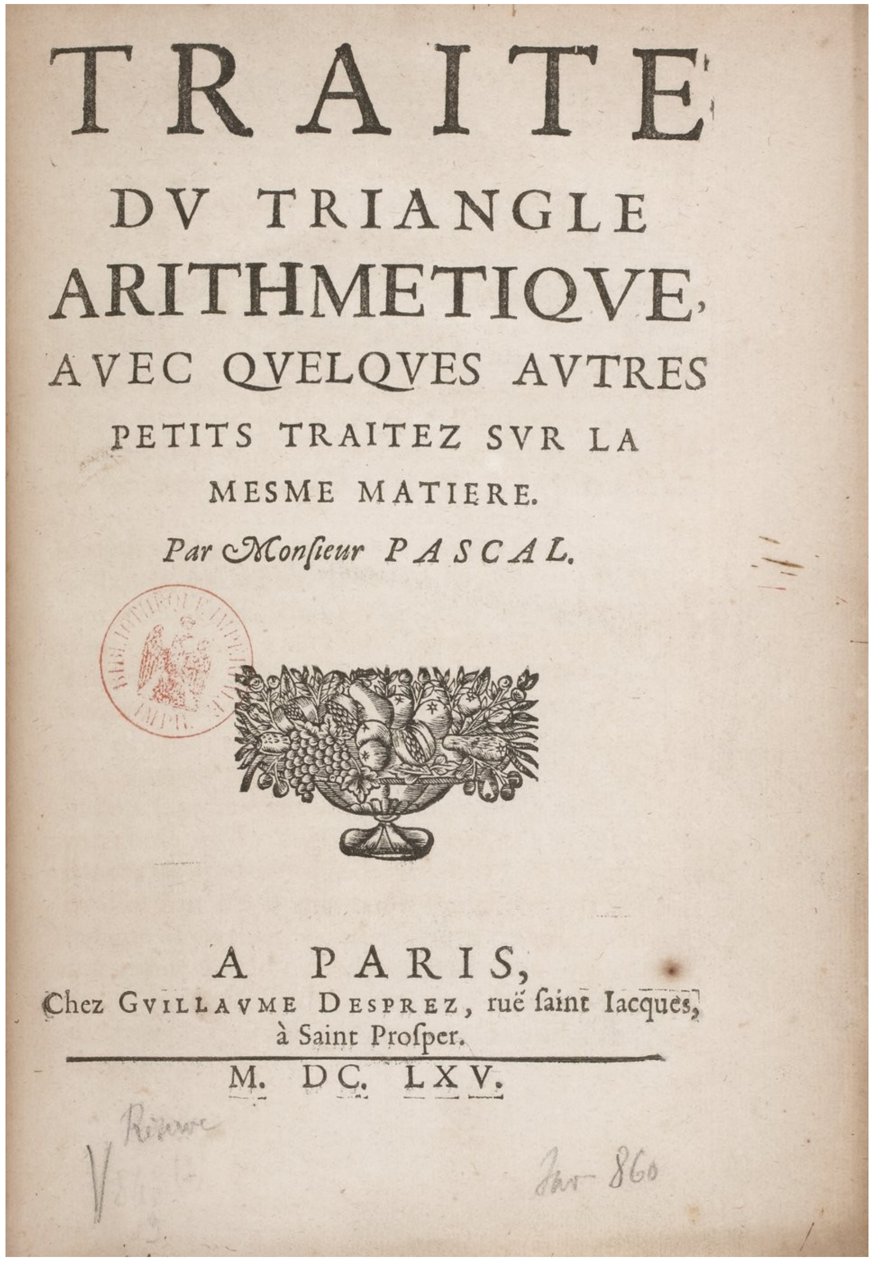
LLLGG
LLGLG
LLGGL
LGLLG
LGLGL
LGGLL
GLLLG
GLLGL
GLGLL
GGLLL

$$
\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&&\\
&&&&&&1&&1&&&&&&\\
&&&&&1&&2&&1&&&&&\\
&&&&1&&3&&3&&1&&&&\\
&&&1&&4&&6&&4&&1&&&\\
&&1&&5&&10&&10&&5&&1&&\\
&1&&6&&15&&20&&15&&6&&1&\\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}
$$

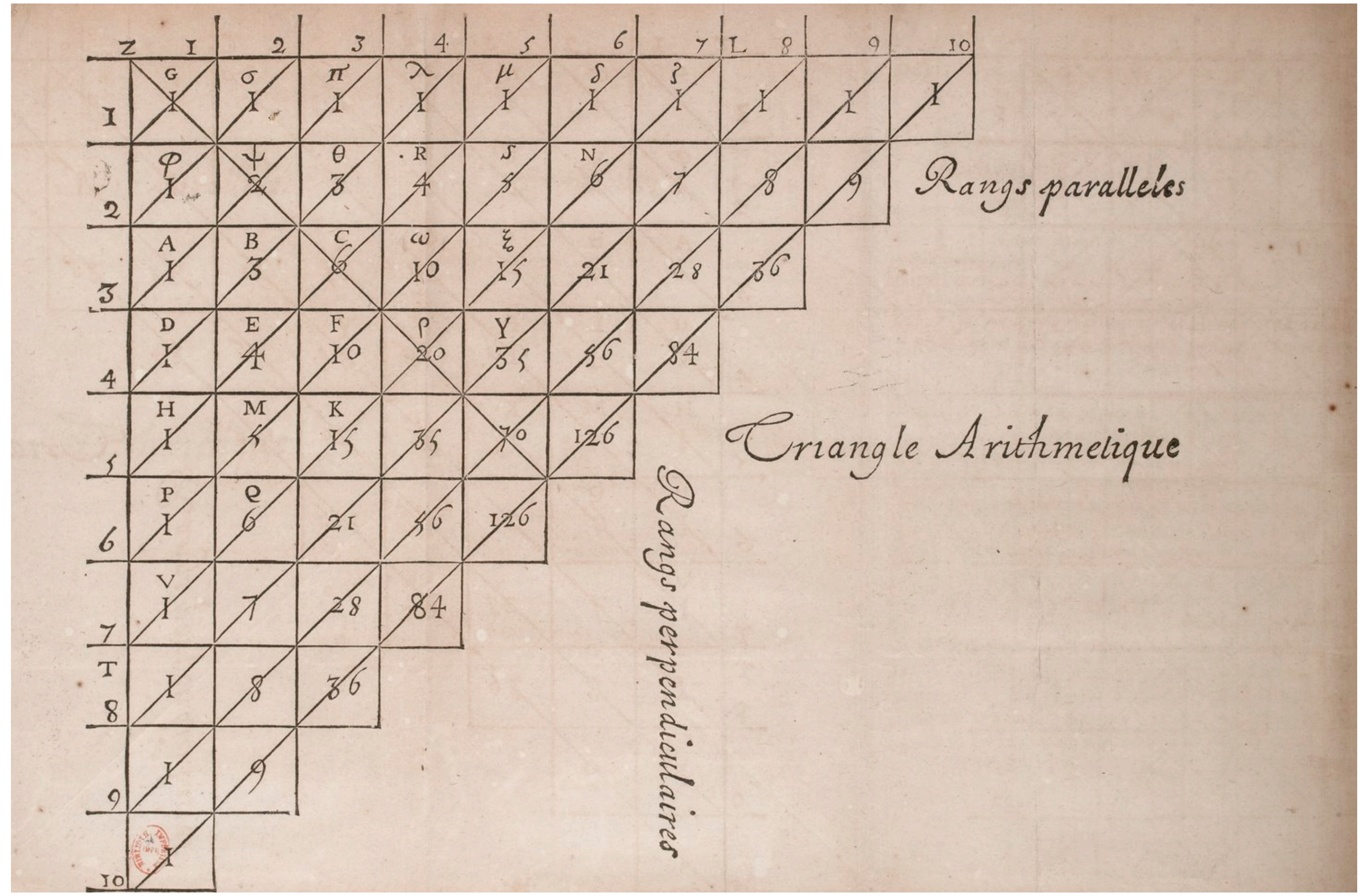
So let’s look on the history of the binomial numbers …

We know that de Moivre was motivated by the problem of tossing a random coin ’n’ times, which in turn was analyzed extensively by Bernoulli, and before by Pascal among others in Europe.

Pascal published in 1653 ‘Traite du triangle arithmetique avec quelques autres’


TRAITE
DV TRIANGLE
ARITHMETIQVE,
AVEC QVELQVES AVTRES
PETITS TRAITEZ SVR LA
MESME MATIERE.
Par Monsieur PASCAL.

A PARIS,
Chez GVILLAVME DESPREZ, ruë saint Iacques,
à Saint Prosper.
M. DC. LXV.



and after that paper we call the triangle of C(n,k) as Pascal’s triangle.

Let us check further….

In 1527 the arithmetic triangle of numbers C(n,k) was published by Petrus Apianus (German scientist working in mathematics, astronomy and cartography)

A first reference in Europe to the triangle of numbers C(n,k) leads to work of Gersonides (Levi Ben Gershon, medieval French-Jewish philosopher) who computed them in publication Maaseh Hoshev (1321)


Eyn Newe
Vnnd wolgegründte
underweysung aller Kauffmanß Rech-
nung in dreyen büchern/mit schönen Re
geln vñ fragstucken begriffen. Sunder-
lich was fortl vnnd behendigkait in der
Welschē Practica vñ Tolletn gebraucht
wirdt/ des gleychen fürmalß wider in
Teütscher noch in Welscher sprach nie
gedrückt. durch Petrum Apianū
von Leyßnick/d Astronomei
zů Ingolstat Ordina-
riū/ verfertiget.

However, apparently binomial C(n,k)

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

can be tracked further down the history.

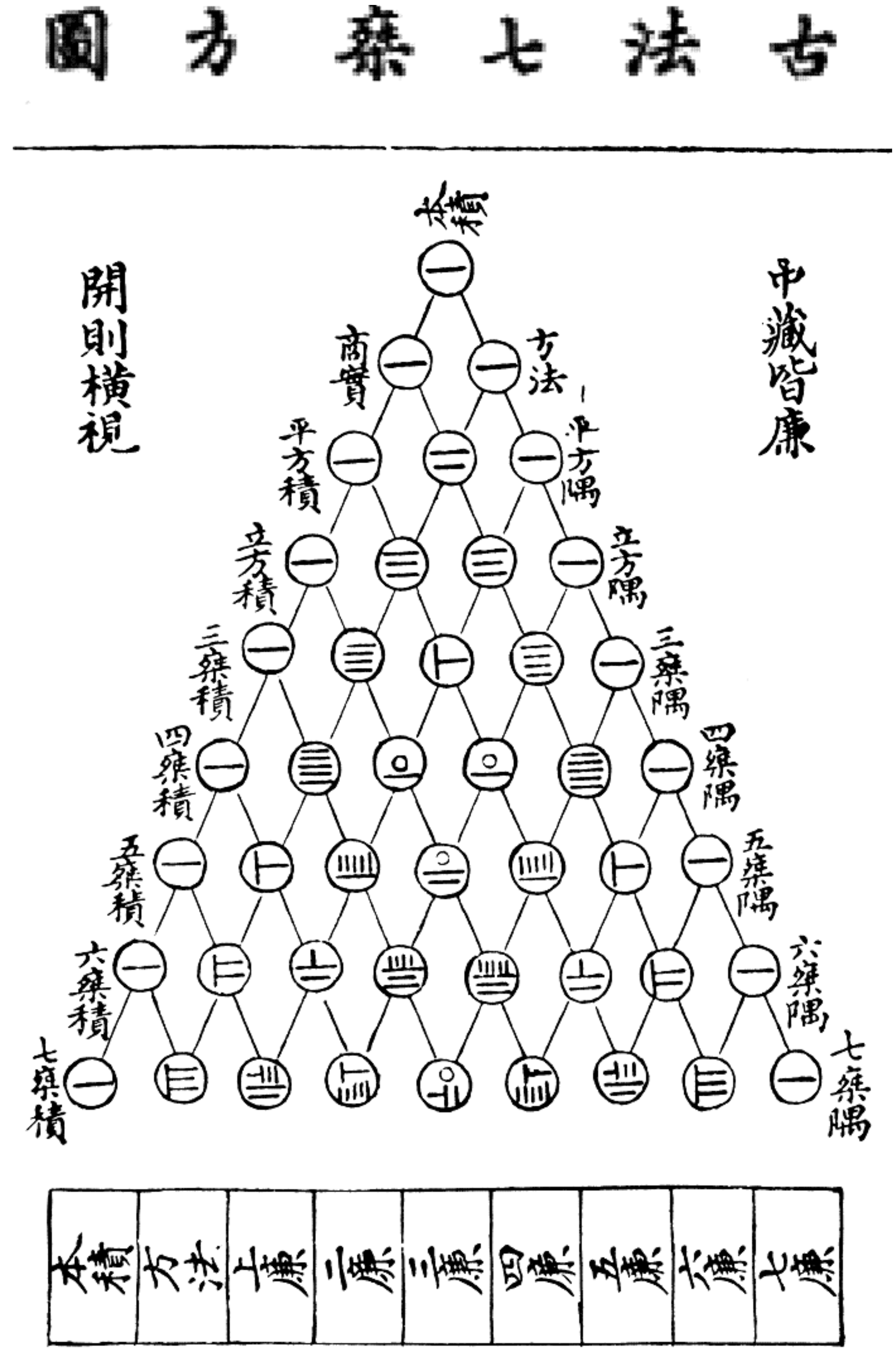
Omar Kayam (1048-1131), a Persian mathematician, astronomer and poet, is claimed to know C(n,k) based on the grounds that he had algorithm to extract n-th roots, and for that you expand $(a+b)^n$.

In fact Omar Kayam refers to Indian mathematicians for algorithm at $n = 2$ and $n=3$, and claims new algorithms for $n>3$.

Let’s check for C(n,k) in the East…

In China the arithmetic triangle of C(n,k) is attributed to Jai Xian (1010-1070) paper “Rújī Shìsuǒ” by mathematician Yang Hui (1238-1298) in his paper “Xiangjie Jiuzhang Suanfa” (1261).

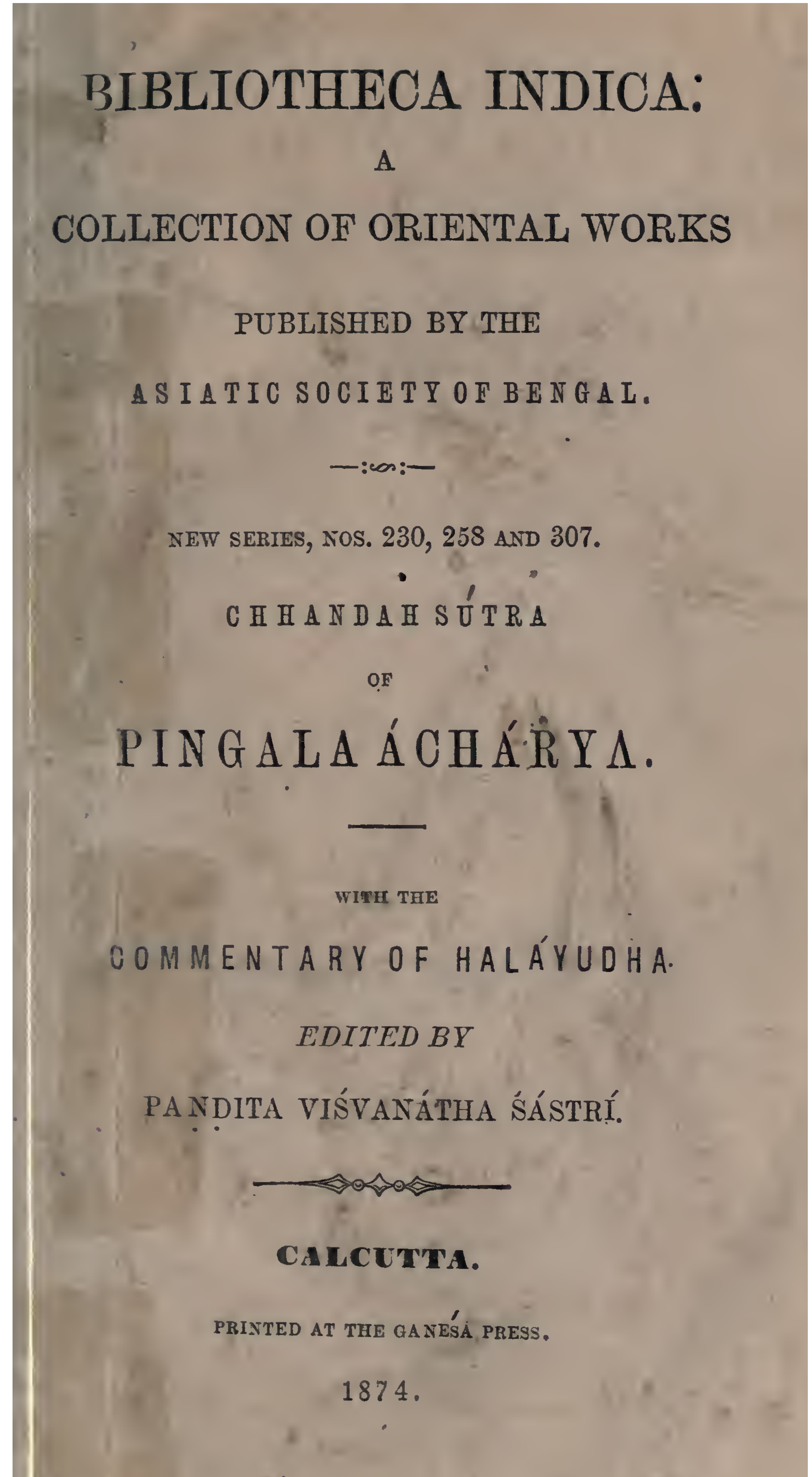
The motivation of Yang Hui and Jai Xian seems to be the same as of Omar Kayam: give algorithms to extract n-th roots using binomial expansion of (a+b)^n



How about India.

That’s where the story
becomes really interesting.

The algorithm to construct the arithmetic triangle C(n,k) by the recursion $C(n,k) = C(n-1,k-1) + C(n-1,k)$ is found in
in the commentary “Mṛtasañjīvanī” written by Halayudha, in 10th century AD about a certain sentence in the paper “Chandaḥśāstra” by Acharya Pingala (circa 200 BC).



BIBLIOTHECA INDICA:

A

COLLECTION OF ORIENTAL WORKS

PUBLISHED BY THE

ASIATIC SOCIETY OF BENGAL.

NEW SERIES, NOS. 230, 258 AND 307.

CHHANDAḤ SÚTRA

OF

PINGALA ÁCHÁRYA.

WITH THE

COMMENTARY OF HALÁYUDHA.

EDITED BY

PAṆḌITA VIŚVANÁTHA ŚÁSTRÍ.

CALCUTTA.

PRINTED AT THE GAṆEŚA PRESS.

1874.

So who is Pingala in 1st-2nd century BC in India and what was the problem he was trying to solve ?

In the modern language Pingala was information theorist worked on the coding theory.

The language of the time was Sanskrit, and substantial portion of the literature was the poetry. Almost all of Sanskrit poetry is based on following of the certain *meter* or arrangement of syllables. *Prosody* is the study of *meter*.

Syllables come come in two equivalence classes (types), an oversimplified model is:

- **light (Laghu), 1 count (1 mātrās): a (अ), i (इ), u (उ), ṛ (ऋ), ḷ (ऌ)**
- **heavy (Guru), 2 count (2 mātrās): ā (आ), ī (ई), ū (ऊ), ṝ (ॠ), e (ए), ai (ऐ), o (ओ), au (औ)**

What is a *meter?*

A meter of n-syllables (aksarachandah) is a binary sequence of length *n of* equivalence classes (L or G) of syllables

For example, 2^n possible aksarachandah of n = 3 syllables are:

LLL	* * *
LLG	* * **
LGL	* ** *
LGG	* ** **
GLL	** * *
GLG	** * **
GGL	** ** *
GGG	** ** **

We show counts (mātrās) in the second column

Why keeping the same meter is useful?

This is error correction code! Composition within a given meter is harder, but memorization and recollection is easier because of embedded error correction code (think as a "check sum").

Therefore, the formal study of the meters (prosody) was important information theory problem (on the coding and the error corrections) and computational linguists such as Pingala have been working on this problem in 100-200 BC in India.

Mathematical equations were formulated as poetry. Precision of oral transmission was very important. Mathematics began to study the mathematical structure of mathematics itself (which was poetry) in 100-200 BC India.

Welcome to recursion!

Several combinatorial problems were addressed by Pingala.

The problem *Lagakriyā* is combination counting.

How many subsets of size *k* in a set of size *n?*

A subset of set X can be encoded by characteristic function from X to {0,1}

Therefore, counting subsets of size *k* of set X is isomorphic to counting {0,1} valued functions on X.

How many binary sequences of length n composed of 'L' and 'G' that have exactly *k* 'L's?

The answer, C(n,k) was found!

Obvious to us. It was research problem at that time.

Some of currently open questions will be embarrassingly obvious a few thousand year in the future?

(१०८) परेण पूर्णम् ॥ १६ ॥

तदेतत्(१)छन्दोवृत्तसङ्ख्याजातं द्विगुणितं पूर्णमेव(२) स्थाप-
यितव्यं न न्यूनं । परे छन्दसि जिज्ञासिते तत् सङ्ख्याजातं
द्विगुणितं परस्य छन्दसो वृत्तानां सङ्ख्या भवति । तद्यथा ।
चतुःषष्टिर्गायत्री समवृत्तानां सङ्ख्या(३) द्विगुणिता सती
परस्योष्णिहः समवृत्तसङ्ख्याष्टाविंशं शतं भवति ॥

(१०९) परे पूर्णमिति ॥ १७ ॥

अतोऽनेकद्वित्रिलघुक्रियासिद्ध्यर्थं यावदभिमतं प्रथमप्रस्ता-
रवत् मेरुप्रस्तारं दर्शयति, उपरिष्टादेकं चतुरस्रकोष्ठं(४)
लिखित्वा तस्याधस्तादुभयतोऽर्द्धनिष्क्रान्तं कोष्ठकद्वयं लिखेत्,

१ तदेवेति पुस्तकान्तरपाठः ।
२ पूर्णमेवेति पुस्तकान्तरपाठः ।
३ समवृत्तसङ्ख्येति पुस्तकान्तरपाठः ।
४ कोष्ठकमिति ख०, पुस्तकान्तरपाठश्च ।

The short scriptures is Pingala's paper (200 BC).

The comments are by Halayudha (around 1000 AD)

The comments contain explicit algorithm of computation of C(n,k) according to translation by Sanskrit experts:

- ***"Ueber die Metrik der Inder"*, Albrecht Weber, Berlin, (1863).**
- ***"Die Pratyayas. Ein Beitrag zur indischen Mathematik"*, Ludwig Alsdorf, Zeitschrift für Indologie und Iranistik, 9, (1933), pp. 97-157**

तस्याप्यधस्तात्त्रयं(१) तस्याप्यधस्ताच्चतुष्टयं यावदाभिमतं स्थानमिति प्रथममेरुप्रस्तारः। तस्य प्रथमे कोष्ठे(२) एकसङ्ख्यां व्यवस्थाप्य लक्षणमिदं प्रवर्त्तयेत्। तत्र परे कोष्ठे यद्वृत्तसङ्ख्याजातं तत् पूर्व्वकोष्ठयोः पूर्णं निवेशयेत। तत्रोभयोः कोष्ठयोरेकैकमङ्कं दद्यात्। ततस्तृतीयायां पङ्क्तौ पर्य्यन्तकोष्ठयोः परकोष्ठगतमेकैकमङ्कं दद्यात्। मध्ये कोष्ठे तु(३) परकोष्ठद्वयाङ्कमेकीकृत्य पूर्णं निवेशयेदिति पूर्णशब्दार्थः। चतुर्थ्यां पङ्क्तावपि पर्य्यन्तकोष्ठयोरेकैकमेव स्थापयेत। मध्यमकोष्ठयोस्तु परकोष्ठद्वयाङ्कमेकीकृत्य पूर्णं त्रिसङ्ख्यारूपं स्थापयेत्। उत्तरत्राप्ययमेव न्यासः।

तत्र(४) द्विकोष्ठायां पङ्क्तौ एकाक्षरस्य विन्यासः। तत्रैकगुर्वेकलघुवृत्तं भवति। तृतीयायां पङ्क्तौ द्व्यक्षरस्य प्रस्तारः। तत्रैकं सर्व्वगुरु द्वे एकलघुनी एकं सर्व्वलघ्विति कोष्ठक्रमेण वृत्तानि भवन्ति। चतुर्थ्यां पङ्क्तौ त्र्यक्षरस्य प्रस्तारः। तत्रैकं सर्व्वगुरु त्रीण्येकलघूनि त्रीणि द्विलघूनि(५) एकं सर्व्वलघु। तथा पञ्चमादिपङ्क्तावपि सर्व्वगुर्व्वादि सर्व्वलघ्वन्तमेकद्व्यादिलघु द्रष्टव्यमिति। षष्ठप्रत्यययोऽप्यर्द्धपरिच्छित्तिरित्येके। सोऽत्यल्प-

१ तस्याधस्तात्त्रयमिति पुस्तकान्तरपाठः।

२ प्रथमकोष्ठके इति ख०, ग० च।

३ मध्यमकोष्ठे तु इति ख०, ग०, पुस्तकान्तरपाठश्च।

४ तत्रापीति पुस्तकान्तरपाठः।

५ त्रीणि त्रीणि एकद्विलघनि इति पुंस्तकान्तर पाठः।

त्वात् पुरुषेच्छानुविधायित्वेनानियतत्वाच्च नोक्तः। एवं प्रत्ययसमासः(१) समाप्तः॥

पिङ्गलाचार्य्यरचिते छन्दःशास्त्रे हलायुधः।
मृतसञ्जीवनीं नाम वृत्तिं निर्म्मितवानिमाम्॥

इति श्रीभट्टहलायुधकृतायां छन्दोवृत्तौ मृतसञ्जीवनीनाम्न्यामष्टमोऽध्यायः समाप्तः॥

म य द्वादश। धीः पञ्चदश। छन्दषोडश। पादः परीक्षिणक् प्रस्तारपङ्क्तिर्विंशतिर्विंशतिरेकविंशतिः। देवतादितोऽष्टौ। चतुःशतं षष्ठो विंशतिर्विंशतिः। युगपरान्तिका त्रयोदश। वृत्तं गावादो विंशतिर्विंशतिः। यवमतौ त्रीणि। यतिर्विंशतिः। वातोर्म्मी मष्टादश। प्रहर्षिणी विंशतिः। शार्दूलविक्रीडितं पञ्चदश। अत्रानुक्तं सप्तदश। इतित्यष्टादशच्छन्दः समाप्तम्॥

१ प्रत्ययविभाग इति पुस्तकान्तरपाठः।

Jayant Shah (Northeastern University) in

"A History of Pingala's Combinatorics" (Ganita Bharati, v. 35, n. 1-2, June-December, 2013)

analyzed Indian literature between Pingala's "Chandaḥśāstra" (about 1st-2nd century BC) and the commentary of Haluyudha in 10th century AD:

c. 2nd century BCE	*Chandaḥśāstra*	*Piṅgala*	Included
2nd century BCE to 1st century CE	*Nāṭyaśāstra*	*Bharata*	Included
c. 550 CE	*Bṛhatsaṃhita*	*Varāhamihira*	Refer to Kusuba
c. 600 CE	*Jānāśrayī Chandovicitiḥ*	*Janāśraya*	Included
c. 7th century CE	*Vṛttajātisamuccaya*	*Virahāṅka*	Included
c. 750 CE	*Pāṭigaṇita*	*Śridhara*	Refer to Shukla
c. 850 CE	*Gaṇitasarasaṅgraha*	*Mahāvira*	Included

Jayant Shah's conclusion: while the preserved evidence from the original text of Pingala is extremely scarce, e.g.

"ekottarakramaśaḥ | pūrvapṛktā lasaṃkhyā"
'Increasing by one, step-by-step, augmented by the next'

the next source "Natyasastra" by Bharata contains better preserved algorithm.

("Natyasastra" is a paper in 4 volumes by Bharata published in 100 BC on theory of danse, music and theater).

Here is algorithm:

ekādhikāṃ tathā saṃkhyāṃ chandaso viniveśya tu |
yāvat pūrñantu pūrveña pūrayeduttaraṃ gañaṃ || (124)
evaṃ kṛtvā tu sarveṣāṃ pareṣāṃ pūrvapūrañaṃ |
kramānnaidhanam ekaikaṃ pratilomaṃ visarjayet || (126)
sarveṣāṃ chandasāmevaṃ laghvakṣaraviniścayam |
jānīta samavṛattānāṃ saṃkhyāṃ saṃkṣepatastathā || (127)

"Natyasastra" by Bharata (2nd-1st century BC)

Jayant Shah's translation:

Put down (a sequence, repeatedly) increased by one up to to the number (of syllables) of the meter.

Also, add the next number to the previous sum until finished.

Also after thus doing (the process of) addition of the next, (that is, formation of partial sums) of all the further (sequences),

1	1	1	1	1
2	3	4	5	6
3	6	10	15	21
4	10	20	35	56
5	15	35	70	126

Next Jayant Shah considers the two algorithms in Virahāṅka (7th century AD):

Algorithm 1: Sūci prastāra:

pramukhente ca ekaikaṃ tathaiva madhya ekamabhyadhikaṃ |
prathamādārabhya vardhante sarvāṇkāḥ || (6.7)
ekaikena bhajyate uparisthitaṃ tathaiva |
paripāṭyā muñcaikaikaṃ sūciprastāre || (6.8)
tatpiṇḍyatāṃ nipuṇaṃ yāvad dvitīyamapyāgataṃ sthānaṃ |
prastārapātagaṇanā laghukriyā labhyate saṃkhyā || (6.9)

Put down the numeral 1, in the beginning, the end and in between (as many as the number of syllables in the meter) and one more. Increase all the numbers starting with the first (as follows.)"

"One-by-one, add the number above (to the partial sum). In the *Sūci* prastāra, successively leave out (the last number) one-by-one."

"The accumulation is complete when the second place is reached (until the number to be left out of addition is in the second place.) *Laghukriyā* number is obtained by carrying out the algorithm."

1	1	1	1	1
1	2	3	4	
1	3	6		
1	4			
1				

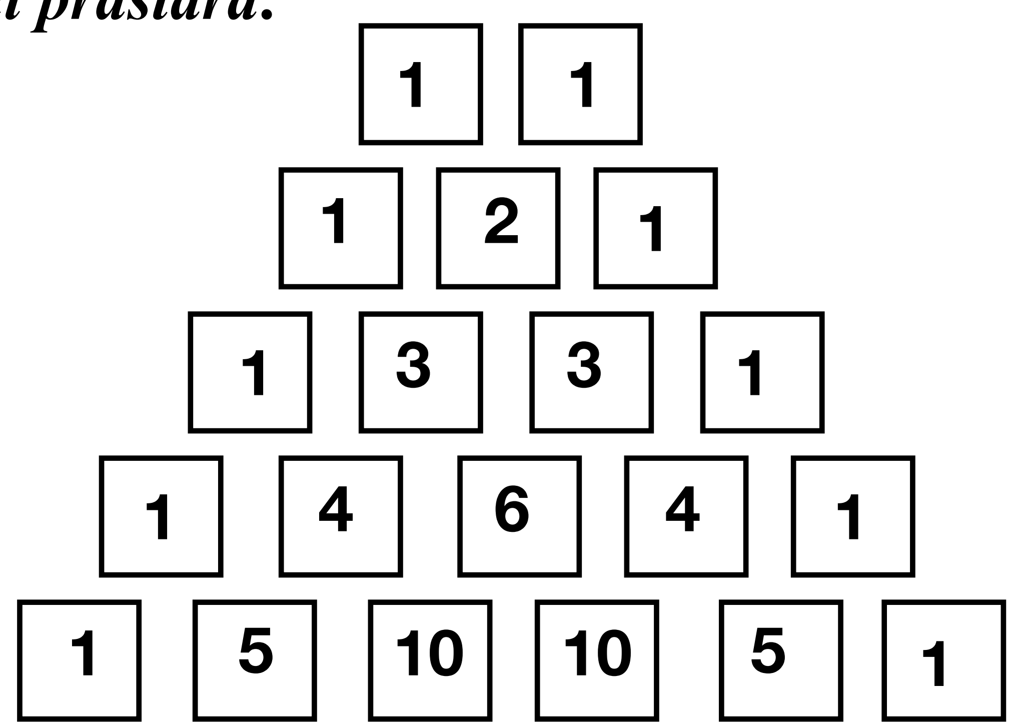
Algorithm 2: Meru prastāra:

iha koṣṭakayordvayorvardhate adhaḥsthitaṃ krameṇaiva |
pramukhānte ekaikaṃ tataśca dvau trayaścatvāraḥ || (6.10)
uparisthitāṅkena vardhate'dhaḥsthitaṃ krameṇaiva |
merau bhavati gaṇanā sūcyā eṣa anuharati || (6.11)
sāgaravarṇe'ṅkau dvaveva gurū madhyamasthāne |
samare punareka eva merau tathaiva sūcyāṃ || (6.12)

(*Meru*) "Two cells (rectangles) in a place, successively increase (the number of cells) below them. In the first and last cell (enter) numeral 1 in (rows) 2, 3, 4 (etc)."

"Step-by-step, in (each) cell below, (place) the sum of the numbers in the (two) cells above. The calculation of the Sūci prastāra is (re)created in the (table called) *meru* (named after the mythical mountain). This (procedure) imitates (it.)"

"In the case of odd number of syllables, there are two large(st) numbers in the middle, moreover, in the case of even number of syllables, there is only one (such) in the *meru*, just as in *Sūci prastāra*."



Of course, both of the algorithms (Sūci prastāra) and (Meru prastāra) are based on the recursion:

$$C(n,k) = C(n-1, k-1) + C(n-1,k)$$

but the order of computation is different.

The first algorithm (Sūci prastāra) generalizes triangular numbers (k=2) to k-symplex numbers.

```
for k from 1 to k_max
        for n from k to n_max
              C(n,k) = C(n-1, k-1) + C(n-1,k)
```

(The case C(n,2) and C(n,3) was known by Greeks,
but Greeks stopped at k=3 because were attached to 3d geometry)

The second algorithm (Meru prastara) is what actually leads to random walk, Gaussian distribution, central limit theorem, Markov processes, heat kernel and finally to the path integral of quantum field theory.

```
for n from 1 to n_max
    for k from 1 to n-1
        C(n,k) = C(n-1, k-1) + C(n-1,k)
```

The recursion

$$C(n,k) = C(n-1, k-1) + C(n-1,k)$$

is a discrete (difference) version of the heat (diffusion) equation, whose solution is the heat kernel in time 'n' and space 'k'. This is one of ways to derive Gaussian from Meru prastaara C(n,k) taking large n.

In continuous limit, if time 'n' is further multiplied by square root of -1 (Wick rotation) we get Schröedinger equation on 1d particle moving on line 'k' and Feynman's formulation of quantum mechanics.

It is wonderful that the first work of information theorists (poets) in India from 200 BC to 700 AD names $C(n,k)$ / discrete Gaussian / heat kernel as *Meru-prastaara* "Mount Meru" which was considered to be the center of all physical, metaphysical and spiritual universes.

While the complexity of the algorithms based on the recursion

$$C(n,k) = C(n-1, k-1) + C(n-1,k)$$

is O(n^2) addition operations for C(n,n/2), at least starting from Mahāvira or Sridhara (800-900 AD) we find algorithm with complexity of O(n) multiplications / division operations:

ekādyekottarataḥ padamūrdhvādharyataḥ kramotkramaśaḥ |
sthāpya pratilomaghnaṃ pratilomaghnena bhājitaṃ sāraṃ |
syāllaghugurukriyeyaṃ saṅkhyā dviguṇaikavarjitā sādhvā

(Write down) the arithmetic sequence starting with one and common difference equal to one upto the number of syllables in the meter above, and in reverse order below (the same sequence). Product of the numbers (first, first two, first three, etc.) (of the sequence) in reverse order divided by the product of the corresponding numbers (of the sequence) in forward order is the *laghukriyā. [Mahāvira, 8th-9th century AD, translated by Jayant Shah, 2013]*

$$\begin{matrix} 1 & 2 \\ 5 & 4 \end{matrix} \; \frac{3*4*5}{3*2*1} = C(5,3) = 10$$

in modern notations the Mahavira/Sridhara algorithm (700 AD) reads as formula

$$C(n,k) = \prod_{i=1}^{k} \frac{n-i+1}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{1 \cdot 2 \dots k}$$

which is a contemporary definition (Newton's) of C(n,k). (By the way, it is applicable when 'n' is not necessarily a positive integer).

Given Maru-prastaara

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

let us see see Boltzmann - Gibbs - Shannon entropy

I will assume engineering perspective (Kolmogorov complexity) which defines *entropy* of a given one time sequence as the binary length of the shortest program (in a language of a fixed expressive power) that generates this sequence.

Remark: it is not difficult to prove that if sequence is sufficiently long, Kolmogorov complexity is not computable. That means, that for a generic compressed sequence, it is not possible to prove that better compression does not exist.

In other words, you never can't exclude that your paper could be made shorter.

Since we can't hope to compute the ideal (theoretical Kolmogorov) complexity of a sequence, let us take practical heuristic approach

Here is an imaginary experiment.

Suppose that Pingala takes a given sequence of light (L) and heavy (G) of total length $n = 4000$.

LLGLLLLGGGGGGGLGLGGGGGGLLGGGGGLLGGGGGLGGGGGGGGGGGLLGGGGGGGGGGGLGGGLGLGLGGGGLLGGGGGGGLGGGGGGGGGLGLGGGGGLGGGGGGGLLLGGGG
GGLGGLLGLGGGGLGGGGGGGGGGGGGGLGLLGGLLLGGLGGGLGGGLLGGGGGGGGGGGGGGGLGLGGGGGGGLGLLGGGGGLLGGGGLGGGGGLGLGGGGGLGGGGLLGGGGGGG
GGGLGGGGGGGGGGGGGGGGGGGGGLLLGLGLGGGGGGGGGGGGGGGLGGLGGGLGGGGLGLLGGGLGGLGGGLGGGGGGGLLGLGGGLLLGLLGGGGGGGGGLGGGGGLGGG
GGGGGGGLLGGGGGGGGGGGGGGGGGGLGGGGLGLGGLGLLGGGGGGGGLLLGGGGGLLGLLGLGGGGLLGGLGGGGLGGGGLGGGGGGLLGGLGGLGLGGLGGGLLLGGGGGGGLLL
GLGGGGGGGLGGGGLGLGGGGGGGGGLLGGGLGLGGGLGGGGGGGLGGGLGGGGGGGGGGLGLGLLGGGGGGGGGLGGLGGGGLGGGGLGLGGLGGLGGGLGGLGGGGLGGGGLGGL
GGGGGLGGGGGLGGGLGLGGGGGGGGGGGGLLGGGGGGGGGLGGGGGLGGLGGLLGGGLGLLGLGGGGLGLGGGGLGLGGGGGLGGGGGGGGGGGGGGGGGGLGGGLGGGGGGGL
GGGGGGGGLGGGGLGGGGGLGLGGGGLLGGGLGGGLGLLGGGGGGGGGGGGGLGGLLGGGLGGGGGGGLGGGLGLLLLGGGGGGGGGGGGGLLGGGLLGLGGGLGGGGGGGGGLGGGG
GLLGGLGGGGGGGGGGGGLGGLLGLGLGLGLLGGGLGGGGLLGGGLLGGGGGLLGLGGLGGLLGLGGLGGLLLGGLGGGLGGGGGGGGGGGGLGGGGGGGLGGGLGGGGLLLGGGGGG
GGGGGGGGGGGGGLGGGGGGGGGLGGGGGLGGGGGGGLGGGGGGLGLGGGGGGLLGGGGGLGGGLGGGGLGLGLLLGLGGGLGGLGLGGGGGGLGGGGGGLGGGGLGGGGGGGLGL
GGGLGGLLLGGLGLGGGGGGLGGGGGGGGGGGLGGGGLGGGGGGGGLGGGGGGLGGGLGGGGGGGGLGGLGLGLGGLGLGGGGGGGGGGGGGGGGGLGLGGGGGGLGGLGGGGGGGL
GGGGGGLGGGGGLGGGGGGGGGLGLLGLGGLGGGGGGGLLLLGGGLGGLLGGLGGLGGLGLGGGGGGGLGGGLGGGLGGGGGGGLGGGGGLLLLGGGGGGLGGGGGGGLGLGLGGGG
GGGGGGGGGLLGLLGGGLGLGLLGGLGGLGLLGLGGGGGLGGLGLGLGGGGGGGGGLGLGGGGGGGLGLLGLGLGLGGGLGGGLGGLGLGGLGGGGLGLGGLLGGLGGLGGGGLLGGGG
LLGGGGGGGGGGGGGGGGGGGLLGGGGLGGGGGLGGGGGGLGGGGGGLGGGGLGGGGGLGLLGLGGLGGLGGGGGGGLGGGLLGLGGLGLGGGGGGGLGGGGGGGGLLGGGGLGLGL
GGGLGGGGLLGGGGGLGGGGGGGGGGLGGGGGGGGGLGGGGLLGLGLGGGLGLLLGGGGGGGLLGLGGGGGGGGGLGGGGLGGLGLGGGGGGGGLLGLGGGLGGGGGGGGGGLGGGLG
LLGLGGGGGGLGLGGGGGGLGGGGGGLGGGGLGGGGGLGGGGGLGGLGGGLGGGGGLGGGGLGGGGGGGLGGGGGLGGGGLLGLGGGGGGGGGGLGGLLLGGGGGGGLGGGLGGLG
GGGGGLGGLGGGGLLGGGGGGGGLGGLGLGGGLGGGGLLGGGGLGGLGGGLGGGGGLGGGGLGGGLGLGLLGGLGGGGGGGGGGGGGGGLGGGGLGLGGGGLGGLGGGGGGLGGGLG
GGLGGLGGGGGGGLGLGLGLGLLGGGGLGLGGGGGGGLLGGGGGGLGGGGLGGLLGGGLGLGGGGGGLLGLGGGGGGGGLLLLGGGGGGGGGLGGGGGGGGGGGLLLGGLGGGGGGG
GGGGGLLGGGGGGGLLGGGGGGGGGLGGGGGGLGGGGGGGGLGLGGGLGGGGGGLLGGGGLGLGGGGGGLGGGLLGGGGGGGGGGGLGLGGGGGGGLGLLGLGGGGGGGLGGGLGGG
GGGGGGGGLGGGGGGGGGGGGLGLLGGGLGGLGLGGGGGGLLGGGGLGGLGLLGGGLGGGGGGLGGLLGGGGGLLGGGGGGGGGGLGLGGGLLLGLGLGLGGLGGGGLLLGGGGGG
LGLGLLGGLLGGGGLGGGGGLGGLGGGGLGGGGGGGGGGGGGGGLGGGGGGGGGLGGGGGGLGGGGGLGGGGGGLGGLGGLGLLGGLGGGGGGGGGGLGLLLLLGLGLGGLGGGGGG
GLGLGGGGGLGLGGLGGLLGGGGLGGLGGGGGLGGGGGLLLGLLLGGGLGLGGGGGGGGGGGGGGGGGGLLGGGGLGGGGGGGGGGGGGGGGGGGLGGGLGLGLGGGGGGGLGGLGG
LGGGLGGGGGGGLGGGLGLGLGLGGGGGGLLLGGGGGGGGGGGGGLLLGGGGGGGGLGGGGGGGGLLGGLGGGGLGGLGGGGGGGGLLGGLGGLGLGGGGGLLLLGGLGGLGGGLLG
LLLGLGGGGGGGGGGGLGGLGGGGGGLGLGLGGGLGGGLGGLLGLGLLGGGLLGGLLLGGGGGLGLGGGGGGGLGGGGLGGGGLGGLGGLGGGGGGGGLGLGGGGGGGGGGGGLGGGG
GGGGGLGLGGLGGGGGLGGGGLGGLGGGGGGGLGGLGGGLLLLGGGGGGGGGGGGGGGGGGGGGLGGLGLLGGLGGGLGGGGGGGGLGGLGLGGGGGLGGGGGLLLGGGGGLGGGG
LLGLGLGGGGGGGGGLGGLLGGGLGLGGGGLLGGGLGGLLLGLGGGGGLGLGGGGGGLGGLGLGGLGGGLGGGGGGGGLGGLGGGGGGGGLGLGGGGGGGGLGGGGGGGLGLGGGLG
GGGLGGGGGGLGLGGLLGLGGLLGLGLLGGGGGGLGGGGGGGGGGGGGGGGGGLGGLGGGGGGGGLGGGGGGLGGGGGGGGGGLGGGLGLGGGLLLGGLGLLGGGLGGGGGGGGGG
GGLLLGLGGLGGGGGGLGLLLGGGLGGLGLGGGGGLGGGGGGLGGGLGGGGGGGGGGGGGLGGLGGGGGGGLGLLGLGGGGGLGGGGLLGLLGGGLGLGGGGLGLGGGGGGGGGGLGG
GGLGGGGLGGGGLLLGGGGGGGGGGLGGGGGGGGGLGLGLGGLLGGGLGGGLLGLGGGGGLGLLGGLLLGGLGGGGGGGGGGGLLGGGGGGGLGGGGLGGLGGGGGGGGGLGLGGLGGG
GGGGGGGGGLGGLGGGGGGGGGLGLLGGGGGLLGGGGGGLLGGLGGGLGGGGGGGGGLGGGLGGLGGGGGGGLLGGGLLGGGGGGGGGGLGGGLGGGGLGGGGGLGLLGGGLGLGG
GGGGGGLLGLGGGLGGGLGGLLLLGGGGGGGGGGGLGGLLGGGGGGLGGGGLLGGGGGGGGGLGGGLGLGGLGGGGGLGGLGGLLGLLLGLGLGGGGGGGGLGGGGGGLGGLGGGGLL
GGGLGGGGGLGGGGGGGGGGGGLGGGGGGGGLLLLGGGGGGLGGGLGGLLGGGGLLLGLGLGGGGLGGGLGGGGGGGGGGGGGGGGGLGGGLGLGGGGGGGGGLGGGGGGGGGLGG
GGGLGGGGGLLGGGLGLGGGGGGLGGGGGLGGGLGGGLLGGGGGGGGGGGGGLGLGLGLGGGGGGGGGLGGGLGLGGGGGGGGGGGGGLGLLGGGGGGLGGGGLGGGLLLGGLGLLGGG
GGGGGLLGGGGGGLLGLLGGLLGGLLLLGGLGGGGGLGGGLLGGLGGGLGGGGGGGLGGGLLGGGGGGGGLGGGLGLGGGGGGGLGGLGGGLGGLGGGGGGGGGLGGGLLGGGGGGLG
GGGLGGGGGLLGLGGGGGLGGLLLGGGLGGGLGLLGGGGGLLGGGGGGLGGGGGGLLGGGGLGGGGLLGGGGGGGLGLLGGGGGGGGGLGGGGLGLGLGGLGGGLGGGGGGGGLLGG
GGGGLGLGGGLGGGGGLGGGGG

and asks how can he encode it efficiently? Pingala might feel that even if he can’t find the best option, he will just try to use some heuristics that he had invented so far.

There 4 main algorithms known to Pingala:

ALG1. From an ordinary number $1 \leq i \leq 2^h$ **produce a binary string at position *i* in the list of all strings of length *h* (in some predermined order, e.g. lexicographic)**

ALG2. Reverse of ALG1

ALG3. From an ordinary number $1 \leq i \leq C(n,k)$ **produce a binary string at position *i* in the list of all strings of length *n* that contain exactly *k* symbols 'L'**

ALG4. Reverse of ALG3

Let us see what is the length of the compressed string if Pingala tries the following encoder:

Step 1. Compute the total length *n* of the sequence and count the number *k* of 'L's.

Result: n = 4000, k = 1000

Step 2. Find the position *i, 1 <= i <= C(n,k)* of a given sequence in the list of all sequences of length *n* with *k 'L' s* [ALG4]

Step 3. Encode the position *i, 1 <= i <= C(n,k)*, to its binary string of length *h* [ALG1]. It is sufficient to use minimal h such that C(n,k) <= 2^h

The decoder works in the reverse way applying ALG2 and then ALG3.

The length of the compressed sequence by Pingala's algorithm is surely

$$h = \lceil \log_2 C(n,k) \rceil$$

$$h = \log_2 C(n,k)$$

In the limit

$$k \gg 1, n \gg 1, k = pn, p = O(1)$$

the length *h* of compressed string is

$$h = \log_2 C(n,k) = \log_2 \frac{n!}{k_1! k_2!}, \quad k_1 = k, \quad k_2 = n - k$$

Using de Moivre-Stirling approximation $\log_2 n! = n \log_2 \frac{n}{e}$

we get

$$h = n \log_2 \frac{n}{e} - \sum_{j=1}^{2} k_j \log_2 \frac{k_j}{e}$$

which is Boltzmann - Shannon - Gibbs entropy formula

$$h = -n \sum_{i=1}^{2} p_i \log_2 p_i, \qquad p_1 = p, \quad p_2 = 1 - p$$

where p_i is frequency of the symbols $p_i = \frac{k_i}{n}$

Let's recapitulate:

- Gaussians integrals are in the heart of quantum field theory

- Their origin is continuous limit of combinatorial objects

- Pingala's compression of a sequence (based only on the total frequency of symbols) and hence on C(n,k) is an approximate simplest upper bound to Kolmogorov's (uncomputable) ultimate entropy. This approximation is called Boltzmann-Gibbs-Shannon entropy.

- Random walk on 1-dimension discrete lattice is computed in Pingala's paper [200 BC], it is called Meru-prastaara C(n,k)

- The continuous limit of Meru-prastaara C(n,k) is Gaussian [paper by de Moivre, 1733].

- We can replace R by Euclidean space R^n without any principal changes. A multivariate Gaussian on R^n is Gauss's 1809 paper on astronomical observations. In the same paper we find linear regression as a maximal likelihood for errors distributed by Gaussian.

- Feynman's reformation of quantum mechanics of a particle on a line R and Schrödinger equation is continuous limit of the Pingala's C(n,k) Meru-prastaara on Z sublattice of R. Extra twist of QM is imaginary time $\sqrt{-1}$

What is next?

The summary was essentially the state of the art circa 1810 about information geometry.

Except that imaginary time came later with quantum mechanics.

The essentially new ideas that appeared after 1810 and continued to the modern mathematics are:

- look on intrinsically non-flat spaces in geometry
- look non-commutative structures in algebra

In geometry, in 1828 Gauss understood 2d surfaces as a 2-dimensional manifold, and in about 1850 Riemann proposed a version of non-flat n-dimensional spaces.

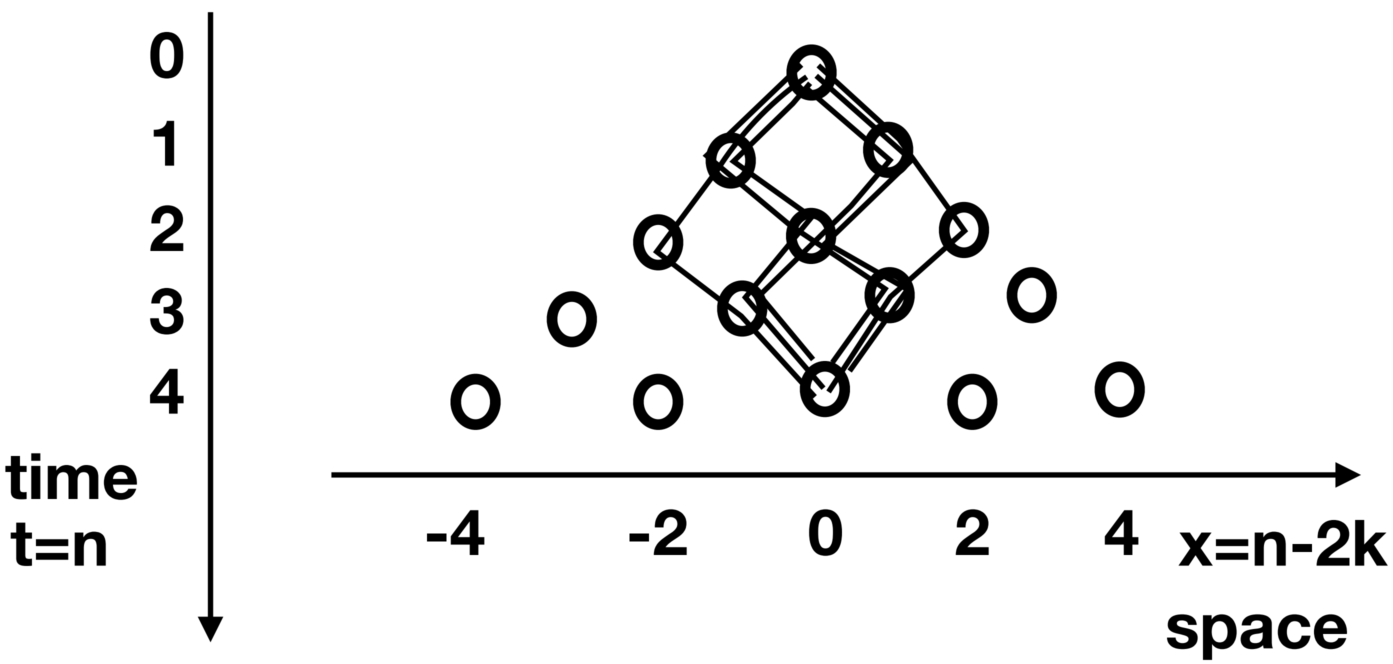
In algebra, in 1820-1830 Galois and Abel started the theory of groups in which multiplication operation was no more necessary commutative.

So what if we combine these new ideas of non-commutative multiplication and non-flat geometry with the random-walk process on the 1-dimensional line obtained by de Moivre from the Pingala's combinatorics of binary sequences?

Recall that in the flat 1-d case, when the domain of random walk is the set of integers, the recursion

$$C(n,k) = C(n-1, k - 1) + C(n-1, k)$$

simply expresses the process in which a particle from position 'k' can move either to the left or to the right



$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Moreover, such classical Pingala-Bernoulli-Pascal-Moivre random walk is commutative! A step is either +1 or -1 on the lattice integer. Then if *s_1, s_2* are steps, we have s1 + s2 = s2 + s1. This commutativity tremendously simplifies the problem of obtaining the probability distribution C(n,k) after *n* steps.

$$C((t_0, x_0), (t_1, x_1)) = \sum_{\text{paths}(t_0, x_0) \to (t_1, x_1))} 1$$

But now imagine that we are studying morally the same Pingala's process of forming sequences of two symbols L and G, but we care also about the order

LG versus GL

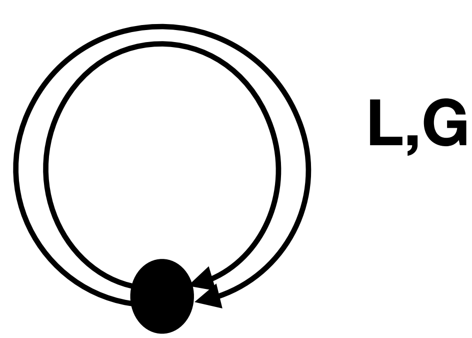
For example, imagine that L and G are consequent operations on something, so we have associativity law:

(L G) L = L (G L)

but not necessarily commutative law:

LG ≠ GL

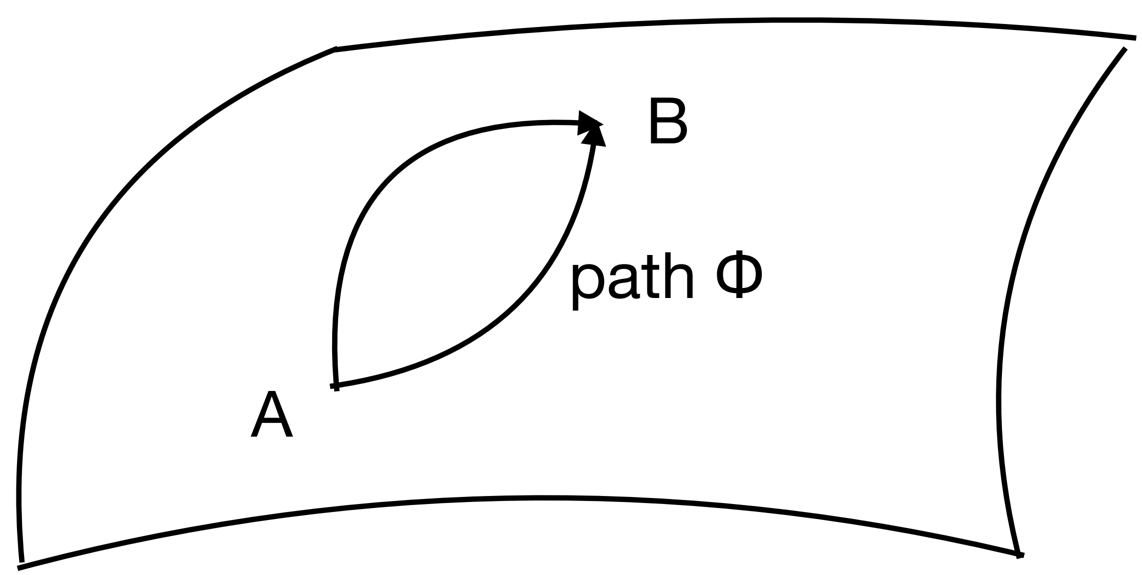
In modern terms we would that the set of sequences formed by L and G is a *monoid* (a category with a single object) generated by two arrows



....LGLLGLGLGGG....

Because of the associativity, the parentheses are not necessary

So, instead of Pingala's (100-200 BC) random walk generated by L = -1 and G=+1 on the flat line where the composition operation is abelian, starting from 19th century we will consider random walks on curved spaces where the order of steps does matter!



In physics this brought non-abelian gauge theory (Yang-Mills theory) and Einstein's general relativity

Anyways, in the first half of 20th century we are still on the same idea:

We are interested in counting PATHS from state A to state B on a space of states X:

$$C(A,B) = \sum_{\text{paths } \Phi : A \to B} \exp(S[\Phi])$$

(except that now X is a curved space unlike Pingala's set of integers)

- **If X is a (pseudo-) Riemannian manifold, Φ is a path from point A to point B, and S[Φ] is the length of the path, the result is Feynman's path integral formulation of quantum mechanics on the space-time X**

- **If X is a group, we get harmonic analysis on groups, very rich topic of 20th century that connected geometry and arithmetics:**

Harish-Chandra —> Langlands Program
—> proof of Fermat's theorem

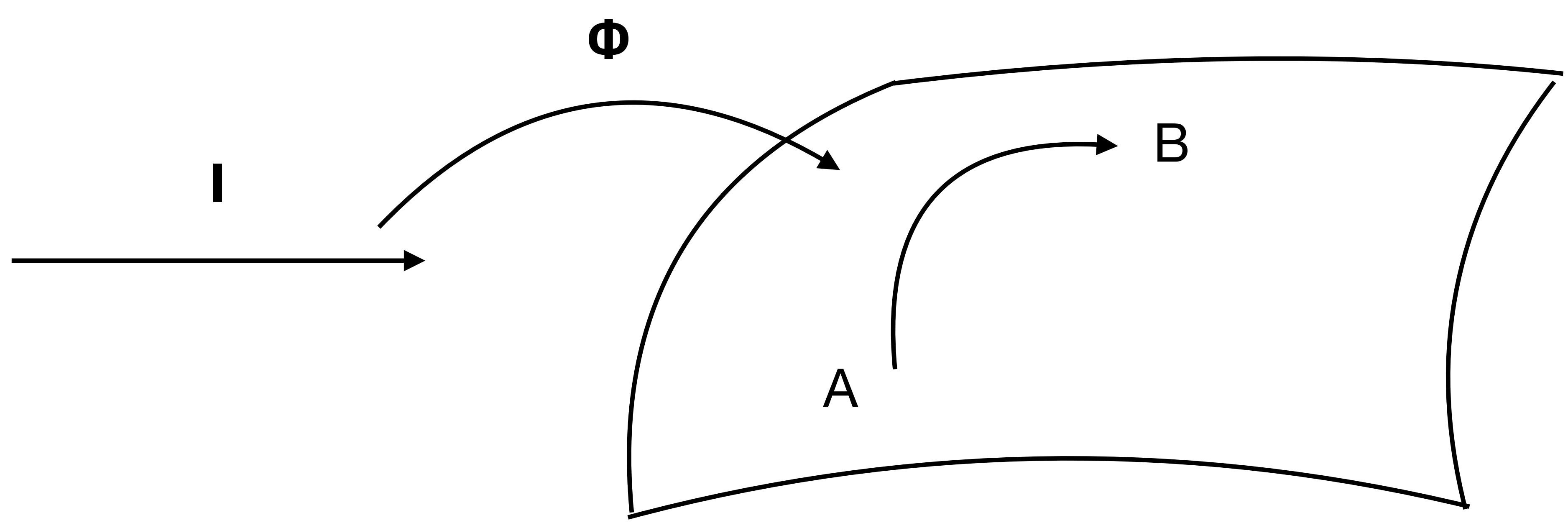
Notice that the sum over 1-d paths from point A to point B on a target X

$$C(A,B) = \sum_{\text{paths } \Phi: A \to B} \exp(S[\Phi])$$

we can write as a

$$C(A,B) = \sum_{\substack{\Phi \in \mathrm{Maps}(I,X) \\ \Phi(\partial I) = \{A,B\}}} \exp(S[\Phi])$$

where the source I is a 1-dimensional interval and the target X is an (n)-dimensional Riemannian manifold S



So in the first half of 20th century the follow-ups on Pingala's paper counted Maps(I,X)

where the source I is 1-dimensional, and the target X is a classical geometrical space (the dimension of X is not as important for complexity)

This gives

- quantum mechanics, harmonic analysis, stochastic processes, Markov chains, probabilistic automata……

What is next?

What gradually happened in the course of the second half of 20th century (and keeps going in the 21st) is the upgrade of the dimension of the source I

Remark: $$\mathrm{Maps}(I, X) = X^I$$

It is much more difficult to increase the dimension of the source I .

if we discretize I and X to size N in every direction, then

$$|X^I| = (N^{d_X})^{N^{d_I}} = N^{d_X N^{d_I}} = \exp(d_X N^{d_I} \log N)$$

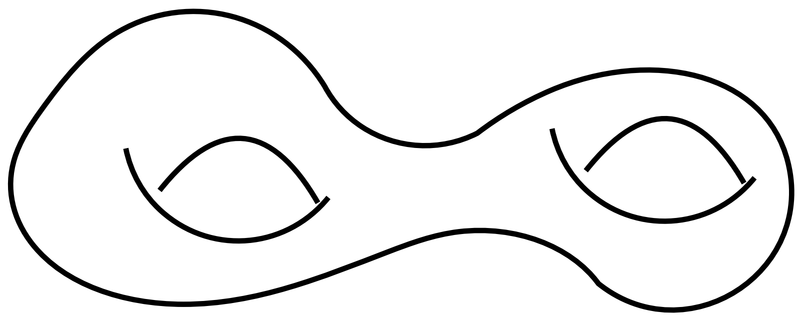
However, keeping the same idea

$$C(A, B) = \sum_{\substack{\Phi \in \mathrm{Maps}(I, X) \\ \Phi(\partial I) = \{A, B\}}} \exp(S[\Phi])$$

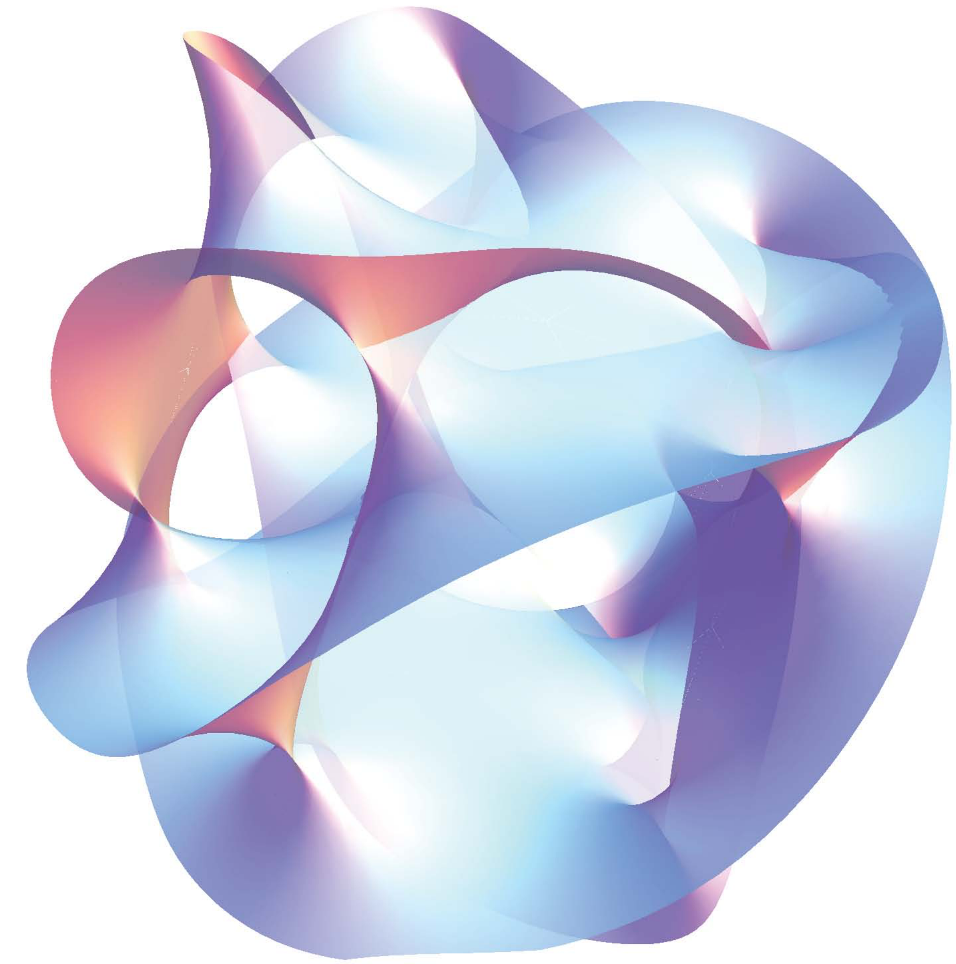
now we take the source *I* to be an n-dimensional manifold!

If the source I is 2-dimensional, the resulting information theory is called *'string theory'*

$$\sum_{\Phi : I \to X}$$



Source space I



Target space X

In a cohomological approximation (after localization) we get 'topological string theory'.

If I has no boundaries, and X is symplectic ==> Gromov-Witten (X)

If I has boundaries, and X is symplectic with extra Lagrangian data ==> Fukaya (X)

We shall not stop at n=2, of course. If go up with the dimension n of the source, the resulting information theory is called

n-dimensional Quantum Field Theory in physics

it is underlying simplified structure in mathematics is n-category

The program of *'string theory'* is to
build the atlas of all possible *interesting* QFTs (in all dimensions)

The information theory (QFT) becomes recursively information theory about information theory about information theory…..

What does it mean *an interesting QFT?*

Recall, that after discretization, a QFT is a distribution on a space of a priori dimension

$$|X^I| = (N^{d_X})^{N^{d_I}} = N^{d_X N^{d_I}} = \exp(d_X N^{d_I} \log N)$$

To describe such QFT naively we would need to write down a string of this length.

An *interesting* QFT is the one whose description we can compress very strongly !

Like in the case of Pingala's compression task, we don't have an algorithm to point out all compressible QFTs (information theories). And as far as we currently understand theory of computational complexity, we will never have a decisive algorithm. The only workable approach so far is heuristic. You hire people and see what interesting QFTs (information theories) they generate.

What are the current lampposts where we are searching for the pages of the atlas of all QFTs?

- Locality: (topological/metric structures)

- Renormalizabilty: sensible limit of large N for $X^I = (N^{d_X})^{N^{d_I}} = N^{d_X N^{d_I}}$

- Various degrees of symmetries (gauge symmetry, supersymmetry, …)

This list is not fixed. Any new organizing principle (with QFTs with short description length) is always welcomed !

With the current lamp-posts we can see *interesting* QFTs or the shadows of them up to the dimension $n = 10$ (11)

There is a conjecture (proofs to the standards of QFT practitioners) that with current lamp-posts, the tower of interesting QFTs terminates at dimension $n = 10$ (11).

So far nothing is found for $n>10$ (11) except *trivial* QFTs.

One of the lamp-posts that keeps under control the naive description length of

$$\exp(N^n)$$

is supersymmetry.

In the first approximation, the idea of supersymmetry is to replace n-dimensional manifold by n|m - dimensional supermanifold.

Locally, tangent to n|m dimensional supermanifold is described by *n* commuting coordinates and *m* anticommuting coordinates, that is Z_2 graded vector space.

A simple example of supermanifold is a total space of a tangent bundle TX with odd (anticommuting) parity in the fiber and even parity in the base, this is called ΠTX.
A function on ΠTX is the same as differential form on X.

$$\phi(x_1, x_2, ..., x_n, dx_1, ..., dx_n) = \sum_{i_1 < i_2 < ... < i_p} \phi_{i_1, ..., i_p} dx_{i_1} \wedge dx_{i_2} \cdots \wedge dx_{i_p}$$

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

These notations invented by Elie Cartan are still in use.
Physicists think about "dx" as a fermion wave-function.

Not all supermanifolds are of the form ΠTX. So the geometry of super-manifolds, is not equivalent in general to the geometry of differential forms on an ordinary manifold, but it is a good first picture to imagine.

There are symmetries. For example, the group of general linear transformations of vector space of dimension n|m is called GL(n|m).

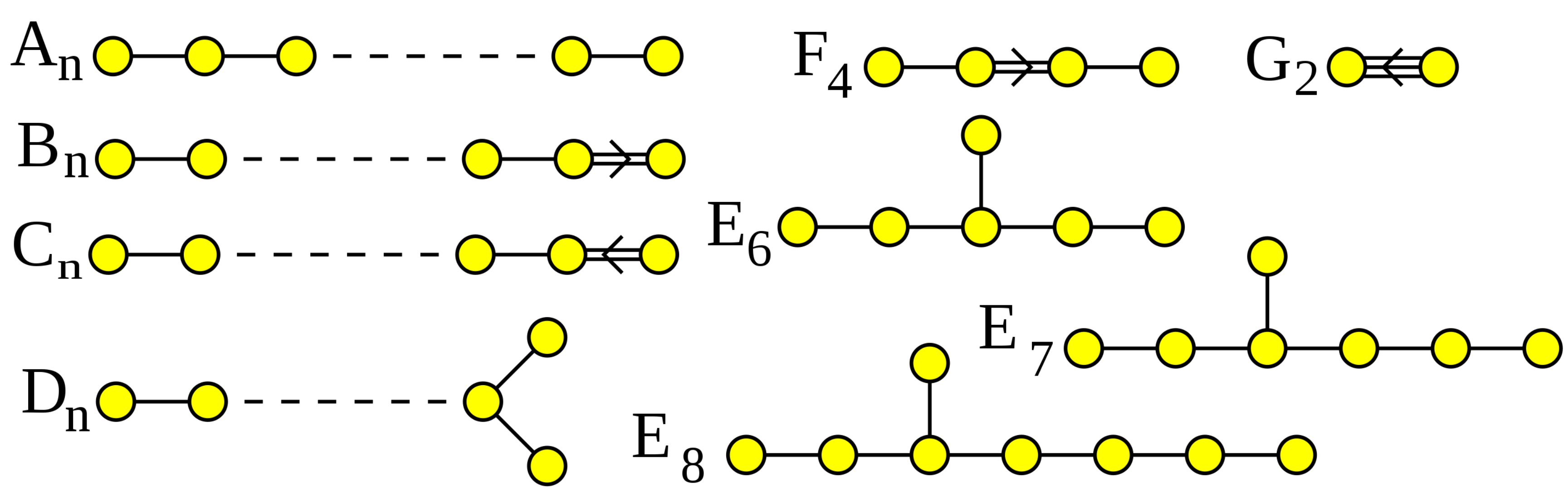
A maximally supersymmetric conformal gauge theory in 4 ordinary commuting dimensions is symmetric under the action of the supergroup PSL(2,2|4); this theory is called

$$\mathcal{N} = 4 \ \mathrm{SYM}$$

The current conjecture of string theory (proved to various degree of certainty) is that we have a complete atlas of irreducible theories in the class N=4 SYM.

The pages of the atlas are labelled by:

- a discrete choice G of a compact simple Lie Group, which was famously classified by Lie, Dynkin and Cartan:



- a modular parameter τ of elliptic curve (a point on complex upper half-plane)

These pages are connected by “transition functors” (dualities are n-functors between QFTs)

$$SYM_{\mathcal{N}=4}(G,\tau) \Leftrightarrow SYM_{\mathcal{N}=4}(G^L, -\frac{1}{n_g \tau})$$

The Langlands dual group G^L comes into the game, which suggests that Langlands functor can be embedded into non-abelian version of Maxwell's duality between electric and magnetic field [Atiyah 1980s, Kapustin-Witten 2003]

The complete mathematical proof of this higher functorial duality

$$SYM_{\mathcal{N}=4}(G,\tau) \Leftrightarrow SYM_{\mathcal{N}=4}(G^L, -\frac{1}{n_g \tau})$$

is not yet achievable by the current techniques. However, there are infinitely many projections of the Left hand side and Right hand side onto something of lower dimension which is computable exactly by localization !

We call such observables the *probes* of QFTs. These supersymmetric probes are the modern versions of measurement tools like LHC. We measure theoretically (compute) some quantities in the left QFT and in the right QFT, and after collecting many evidences that the measurements (projections) coincide we think that a given pair of QFTs is isomorphic.

The main technique is Atiyah-Bott equivariant localization formula applied to (infinite-dimensional) functional spaces of Maps(I,X).

The idea of the corresponding mathematics of equivariant cohomology was very well explained in A. Alekseev talk on Monday, and during the panel session, so I'll not repeat.

The localisation formula for a Lie group T acting on a manifold $\mathcal{X}$ reads

$$\int_{\mathcal{X}} e^{-S} = \int_{\mathcal{X}^T} \frac{i^* e^{-S}}{e_T(N_{\mathcal{X}^T})}$$

In case of the 4d SYM:

1) $\mathcal{X}$ is (roughly) infinite-dimensional space of the fields of the SYM, roughly it is a Dirak determinant bundle over Maps(M_4, BG) where M_4 is 4-dimensional space time, and BG is classifying space of G, e.g. the space of G-bundles on M_4 with connection

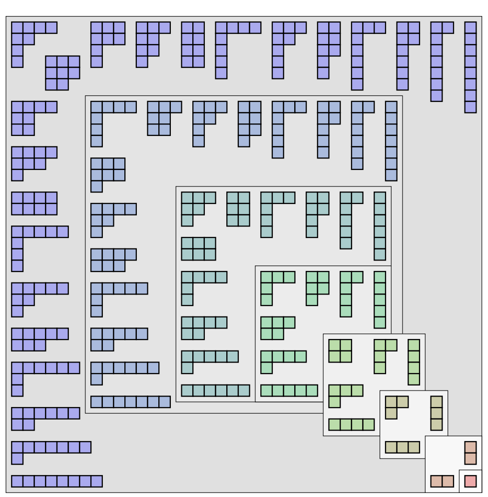
2) the equivariant Euler classes (determinants) are replaced by equivariant superEuler classes (super-determinants)

We get infinities under control in this way.
Typical expressions which come out from the infinite-dimensional determinants are simple infinite products like

$$\Gamma_{\epsilon_1,\epsilon_2,\epsilon_d}(x) = \prod_{n_1,n_2,..,n_d\geq 0} (x + n_1\epsilon_1 + n_2\epsilon_2 + \ldots n_d\epsilon_d)$$

which is a version of multi-dimensional Gamma function found by Barnes in 1899, and the determinants like that are summed over fixed points labelled by d-dimensional partitions. For d=2 is it like in Euler's function

$$Z(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots = \prod_{n=1}^{\infty}(1-q^n)^{-1}$$



We get a non-trivial partition function $\mathcal{Z}$

of many variables (omitted in this talk)
of the same algorithmic complexity class as
- generating function of Gromov-Witten invariants of arbitrary genus in toric Calabi-Yau three-folds
- correlation functions of the 2d CFT

This function $\mathcal{Z}$ serves to check transformations between pages of atlas relating

dual quantum field theories like Maxwell-Langlands modular transform.

Another example of famous duality between 2d QFTs is called Mirror Symmetry.

Mirror symmetry of string theory relates a pair of QFTs

A-theory(symplectic target X) <———-> B-theory(complex target Y)

A-theory (X) B-theory (Y)

X is symplectic manifold Y is complex manifold

$\mathrm{Maps}_A(I, X)$ $\mathrm{Maps}_B(I, X)$

A-branes <———> B-branes

Mirror functor: Fukaya Category(X) <———> D(Coh(Y))

Z[Gromov-Witten invariants] <———> Z[Periods]

Number of rational curves of degree k in quintic Calabi-Yau (degree 5) three-dimensional hypersurface in P4

string theory localization 1990

“conventional” algebraic geometry 1993

Nuclear Physics B359 (1991) 21–74
North-Holland

A PAIR OF CALABI–YAU MANIFOLDS AS AN EXACTLY SOLUBLE SUPERCONFORMAL THEORY*

Philip CANDELAS[1], Xenia C. DE LA OSSA[1,**], Paul S. GREEN[2] and Linda PARKES[1]

TABLE 4
The numbers of rational curves of degree k for $1 \leqslant k \leqslant 10$

k	n_k
1	2875
2	6 09250
3	3172 06375
4	24 24675 30000
5	22930 58888 87625
6	248 24974 21180 22000
7	2 95091 05057 08456 59250
8	3756 32160 93747 66035 50000
9	50 38405 10416 98524 36451 06250
10	70428 81649 78454 68611 34882 49750

number of conics [28] (rational curves of degree two). Clemens has shown [30] that $n_k \neq 0$ for infinitely many k and has conjectured that $n_k \neq 0$ for all k, but it seems that the direct calculation of these numbers becomes difficult beyond $k = 2$ (see also ref. [28]). It is however straightforward to develop the series (5.12) to more terms and to find the n_k by comparison with (5.13). We present the first few n_k in table 4. These numbers provide compelling evidence that our assumption about

https://arxiv.org/abs/alg-geom/9301006v2

Rational curves on Calabi-Yau manifolds: verifying predictions of Mirror Symmetry

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Stillwater, OK 74078
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Recently, mirror symmetry, a phenomenon in superstring theory, has been used to give tentative calculations of several numbers in algebraic geometry [1]. This yields predictions for the number of rational curves of any degree d on general Calabi-Yau hypersurfaces in $\mathbf{P}^4$ [2], $\mathbf{P}(2,1^4)$, $\mathbf{P}(4,1^4)$, and $\mathbf{P}(5,2,1^3)$ [4, 9, 11]. The techniques used in the calculation rely on manipulations of path integrals which have not yet been put on a rigorous mathematical footing. On the other hand, there is currently no prospect of calculating most of these numbers by algebraic geometry.

Until this point, three of these numbers have been verified, all for the quintic hypersurface in $\mathbf{P}^4$: the number of lines (2875) was known classically, the number of conics (609250) was calculated in [7], and the number of twisted cubics (317206375) was found recently by Ellingsrud and Strømme [3].

For a QFT with moduli <u>θ</u>

$$Z = \int D\Phi e^{-S[\Phi,\theta]}$$

from the variation

$$\tilde{S} = S + \delta_\theta S$$

define

$$g(\delta_\theta S, \delta_\theta S) = \langle \delta_\theta S \delta_\theta S \rangle - \langle \delta_\theta S \rangle \langle \delta_\theta S \rangle$$

in physics this is simply called the natural metric on the moduli space of QFTs. (For 2d CFTs it is called in particular Zamolodchikov's metric).

Of course it she same formula which is called Fischer's metric in statistics. Shall we call it Pingala's metric?

The geometry of moduli spaces is a very rich topic in physics and mathematics. If QFT has extra geometrical structures (supersymmetry), the moduli space comes with natural extra geometrical data (Kahler, special Kahler, quaternionic Kahler, hyperKahler, etc)

For example, Jockers. et al in https://arxiv.org/abs/1208.6244v3

computed Kahler metric on the moduli of 2d supersymmetric QFTs on a two-sphere. by localization. For a particular QFT flowing to sigma-model on a quintic, they compute easily all genus 0 GW invariants on a 3d quintic.

$$Z_{\text{quintic}} = (z\bar{z})^{\mathsf{q}} \oint \frac{d\epsilon}{2\pi i}(z\bar{z})^{-\epsilon} \; \frac{\pi^4 \sin(5\pi\epsilon)}{\sin^5(\pi\epsilon)} \left| \sum_{k=0}^{\infty} (-z)^k \, \frac{\Gamma(1+5k-5\epsilon)}{\Gamma(1+k-\epsilon)^5} \right|^2 \; .$$

Here ‘k’ labels fixed points in Atiyah-Bott localization, and 1-Gamma() comes from infinite determinants of quantum fields in 2d.

For all fancy details of other localization use in QFT see 700 pages review :

Localization techniques in quantum field theories

Vasily Pestun, Maxim Zabzine,
Francesco Benini, Tudor Dimofte,
Thomas T. Dumitrescu,
Kazuo Hosomichi, Seok Kim,
Kimyeong Lee, Bruno Le Floch,
Marcos Marino, Joseph A. Minahan,
David R. Morrison, Sara Pasquetti,
Jian Qiu, Leonardo Rastelli,
Shlomo S. Razamat, Silvu S. Pufu,
Yuji Tachikawa, Brian Willett,
Konstantin Zarembo

review collection
arXiv:1608.02952

to be continued…

Thank you all