Holomorphic Jacobi manifolds and their global counterparts

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Introduction

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- A natural bridge between these two geometric frameworks is Jacobi geometry.
- This talk will focus on Poisson stuctures in relation with Jacobi structures. Recall that Jacobi manifolds were introduced independently by Lichnerowicz (1978) and Kirillov (1976) who used different but equivalent definitions.

Goals

- A Jacobi structure on a smooth manifold M is a Lie bracket on its algebra of smooth functions $C^\infty(M, \mathbb{R})$. Jacobi structures were introduced as generalizations of Poisson structures. But looking Jacobi structures from a certain viewpoint, we see that they are just richer versions of Poisson structures. Our goal is twofold:

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  1. First, we will explain the following diagram:

Jacobi manifolds	$\Longleftarrow$	holomorphic Jacobi	$\Longrightarrow$	generalized contact
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
homogeneous Poisson manifolds	$\Longleftarrow$	homogeneous holomorphic Poisson	$\Longrightarrow$	homogeneous gen. complex

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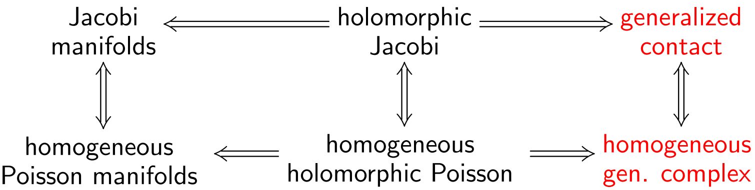

  2. Second, we will explain the following result:

Main theorem

Theorem: Integrable holomorphic Jacobi manifolds are in one-to-one correspondence with source-simply connected contact groupoids, i.e. complex groupoids equipped with a multiplicative complex contact structure.

Jacobi manifolds

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- A contact structure on a smooth manifold M is defined by a co-dimension 1 maximally non-integrable distribution $\xi \subseteq TM$. Let $L := TM/\xi$ be its associated line bundle. The distribution ξ can be equivalently defined by a line bundle-valued 1-form $\Theta : TM \to L$ which is viewed as the canonical projection. The 1-form Θ defines a non-degenerate Jacobi structure $J : \Lambda^2(\mathfrak{J}^1 L) \to L$.

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Jacobi manifolds

- A Jacobi structure on M is given by a line bundle $L \to M$ and a Lie bracket $\{\cdot,\cdot\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ which is a bi-derivation, that is a derivation with respect to each entry.

Definition: By a derivation of L, we mean an $\mathbb{R}$-linear operation $\Delta : \Gamma(L) \to \Gamma(L)$ satisfying:

$$\Delta(fe) = f\Delta(e) + (\sigma(\Delta) \cdot f)e,$$

where $\sigma(\Delta)$ is the symbol of Δ. Derivations of L can be identified with infinitesimal isomorphisms of L.

The gauge Lie algebroid and Jacobi brackets

In other words, derivations of L are sections of the Lie algebroid $DL \to M$ called the gauge (or Atiyah) Lie algebroid of L. Its anchor map is the symbol and its Lie bracket is the commutator of derivations and let $\mathfrak{J}^1 L$ be the first jet bundle of L. We have the vector bundle isomorphisms:

$$DL \simeq \mathrm{Hom}(\mathfrak{J}^1 L, L) \quad \text{and} \quad \mathfrak{J}^1 L \simeq \mathrm{Hom}(DL, L).$$

Remark: Given a Jacobi manifold $(M, L, \{,\cdot,\cdot\})$, there is an associated 2-form $J : \Gamma(\Lambda^2(\mathfrak{J}^1 L)) \to \Gamma(L)$ defined by:

$$\{\lambda, \mu\} = J(j^1\lambda, j^1\mu),$$

for all $\lambda, \mu \in \Gamma(L)$.

Atiyah forms

- Consider the de Rham complex of DL with coefficients in L, denoted $(\Omega_L^\bullet := \Gamma(\Lambda^\bullet(DL)^* \otimes L), d_{DL})$
- Cochains in $(\Omega_L^\bullet, d_{DL})$ are called Atiyah forms.

Contact structures: Consider a contact manifold (M, ξ) with its associated line bundle $L := TM/\xi$. Let $\omega = d_{DL}\Theta$, where $\Theta : TM \to L$ is the canonical projection. Then, ω is a non-degenerate Atiyah 2-form, i.e. the bundle map $\omega^\flat : DL \to \mathfrak{J}^1 L$ is invertible. We denote its inverse by $J^\sharp : \mathfrak{J}^1 L \to DL$.

Contact geometry revisited

Let M be an odd-dimensional manifold with a fixed line bundle L.

Proposition: Contact structures $\xi \subseteq TM$ with associated line bundle $L = TM/\xi$ are in one-to one correspondence with non-degenerate d_{DL}-closed Atiyah 2-forms on L.

We have the correspondence: $\xi \mapsto \omega = d_{DL}(\Theta \circ \sigma)$, where $\Theta : TM \to TM/\xi$ is the canonical projection giving a Jacobi tensor $J := \omega^{-1} : \Lambda^2 \mathfrak{J}^1 L \to L$. Conversely, every non-degenerate Jacobi structure on L corresponds to a contact structure on (M, L).

Other Basic Examples

1. Any Poisson manifold (M, π) is a Jacobi manifold on the trivial line bundle $L = M \times \mathbb{R}$. The first jet bundle of L is $\mathfrak{J}^1 L = T^*M \oplus \mathbb{R}$. The associated 2-form $J : \Lambda^2(\mathfrak{J}^1 L) \to L$ can be written in the matrix form: $J_\pi = \begin{pmatrix} \pi & 0 \\ 0 & 0 \end{pmatrix}$.
2. Let $\mathfrak{g}$ be a real Lie algebra. Consider
   - $M = \mathbb{RP}(\mathfrak{g}^*)$ the projective space.
   - $L \to M$ the tautological space.
   - There is an inclusion $\iota : \mathfrak{g} \hookrightarrow \Gamma(L)$ defined by:

     $$\iota(v) = \lambda_v, \quad \text{with} \quad \lambda_v(r) = \ell_v|_r, \quad \forall r \subseteq \mathfrak{g}^*$$

     where $\ell_v : \mathfrak{g} \to \mathbb{R}$ is the linear function corresponding to v.
   - In this case, the Jacobi bracket on $\Gamma(L)$ is given by

     $$J(\lambda_v, \lambda_w) = \lambda_{[v,w]}.$$

Homogenization scheme

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- Every section $\lambda \in \Gamma(L)$ can be identified with a fiber-wise linear function on L^*. By restriction, it can be considered as a homogeneous function de degree one on $\widetilde{M}$, denoted $\widetilde{\lambda}$. The correspondence $\lambda \mapsto \widetilde{\lambda}$ is one-to-one. Furthermore there is a one-to-one correspondence between Atiyah forms in $\Omega_L^\bullet := \Gamma(\Lambda^\bullet(DL)^* \otimes L$ and homogeneous forms on $\widetilde{M}$.

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Homogenization scheme

- In particular, if $\xi \subseteq TM$ defines a contact structure on M and ω denotes its associated symplectic Atiyah 2-form, then the homogenization $\widetilde{\omega}$ of ω is a homogeneous symplectic form on $\widetilde{M}$, i.e. $\mathcal{L}_Z \widetilde{\omega} = \widetilde{\omega}$, where Z is the Euler vector field on $\widetilde{M}$. More generally, we have the following result:

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Proposition: Jacobi stuctures on (M, L) are in one-to-one correspondence with homogeneous Poisson brackets on $\widetilde{M} = L^* \setminus \{0_M\}$.

Holomorphic Jacobi manifolds

- Let $X = (M, j)$ be a complex manifold together with a holomorphic line bundle $L \to X$. By a holomorphic Jacobi structure on (X, L), we mean a Lie bracket on the sheaf of holomorphic sections of L which is a first order bi-differential operator.

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- Holomorphic Jacobi structures on (X, L) are equivalent to holomorphic sections $J \in \Gamma(\Lambda^2(\mathfrak{J}^1 L)^* \otimes L)$ satisfying some integrability conditions.

Holomorphic Poisson manifolds

Reminder: Let $X = (M, j)$ be a complex manifold. A holomorphic Poisson structure on X is given by a Poisson bracket $\{-,-,\}$ on the sheaf of holomorphic functions on X.This is equivalent to holomorphic Poisson bivector $\Pi \in \Gamma(\Lambda^2 T^{(1,0)}X)$ which satisfies $\overline{\partial}\Pi = 0$ and $[\Pi, \Pi] = 0$.

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Remark:

1. Holomorphic Poisson manifolds are as special cases of holomorphic Jacobi manifolds that were extensively studied. See for instance, papers by Gualtieri-Bailey (2017); Chen, Fino and Poon (2017); Hitchin (2012); Fiorenza and Manneti (2012); Laurent-Gengoux, Stienon, and Xu (2008), etc.
2. Any holomorphic Poisson manifold (X, Π) comes with a holomorphic Lie algebroid $(T^*X)_\Pi$.

Examples of holomorphic Jacobi manifolds

- Holomorphic vector fields on complex manifolds
- Holomorphic contact structures
- The dual $\mathfrak{g}^*$ of a Lie algebra endowed with a 1-cocycle.
- Projective space $\mathbb{CP}(\mathfrak{g}^*)$ of the dual of a complex Lie algebra,

Generalized contact structures

Definition: A generalized contact structure on M is a line bundle $L \to M$ together with an endomorphism $\mathcal{I} : \mathbb{D}L = DL \oplus \mathcal{J}^1 L \to \mathbb{D}L$ satisfying:

1. $\mathcal{I}^2 = -\mathrm{id}$
2. $\mathcal{I}$ is skew symmetric with respect to the L-valued symmetric pairing $\langle\langle -, - \rangle\rangle$ on $\Gamma(\mathbb{D}L)$ defined by: $\langle\langle (\Delta_1, \Psi_1), (\Delta_2, \Psi_2) \rangle\rangle := \langle \Delta_1, \Psi_2 \rangle_L + \langle \Delta_2, \Psi_1 \rangle_L$,
3. the Nijenhuis torsion of $\mathcal{I}$ is zero, i.e. $[\![\mathcal{I}\alpha_1, \mathcal{I}\alpha_2]\!] - [\![\alpha_1, \alpha_2]\!] - \mathcal{I}[\![\mathcal{I}\alpha_1, \alpha_2]\!] - \mathcal{I}[\![\alpha_1, \mathcal{I}\alpha_2]\!] = 0$, for all $\alpha_i = (\Delta_i, \Psi_i) \in \Gamma(\mathbb{D}L)$, where the bracket $[\![-, -]\!]$ is given by: $[\![(\Delta_1, \Psi_1), (\Delta_2, \Psi_2)]\!] = ([\Delta_1, \Delta_2],\ \mathcal{L}_{\Delta_1}\Psi_2 - \iota_{\Delta_2} d_{DL} \Psi_1)$

Holomorphic Poisson manifolds

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- Holomorphic Poisson structures were studied within the context of generalized complex geometry by Laurent-Gengoux, Stienon, and Xu who proved:

Theorem (Laurent-Gengoux, Stienon, Xu, 2008):
Any holomorphic Poisson structure Π on $X = (M, j)$ determines a generalized complex structure $\mathbb{J} := \begin{pmatrix} j & (\Pi)^\sharp_{\mathrm{Re}} \\ 0 & -j^* \end{pmatrix}$, where $(\Pi)_{\mathrm{Re}}$ is the real part of Π.

Holomorphic Poisson manifolds

- Recall that any holomorphic Poisson manifold (X, Π) determines a holomorphic Lie algebroid T^*X_Π. In fact, Laurent-Gengoux, Stienon, and Xu proved that:

$$(T^*X_\Pi)_{\mathrm{Re}} \simeq (T^*M)_{4\Pi_{\mathrm{Re}}}$$

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- More generally, a holomorphic Lie algebroid is integrable if and only if its underlying real Lie algebroid is integrable.
- Moreover, holomorphic Poisson manifolds integrate into complex symplectic groupoids, i.e. a complex groupoid with a compatible multiplicative complex symplectic 2-form.

Holomorphic Jacobi and generalized contact structures

- A complex manifold $X = (M, j)$ cannot admit a generalized contact structure for dimension reasons! Moreover, any holomorphic line bundle $L \to X$ is a rank 2 vector bundle over M. So, a direct homogenization scheme won't work!
- But we noticed that, in the complex, we can perform the following two-step procedure:
  1. First we pass from a holomorphic line bundle $L \to X$ to a real $U(1)$-principal bundle $\widehat{M} \to M$ equipped with a canonical real line bundle $\widehat{L}$ by taking $\widehat{M} = \mathbb{RP}(L^*)$ (which is obviously of odd dimension) and the dual of the tautological bundle $\widehat{L} \to \widehat{M}$
  2. Check that the homogenization of $\widehat{L} \to \widehat{M}$ agrees with $\widetilde{X} = L^* \setminus \{O_M\}$.

We keep track of the complex structure at all steps.

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We proved:

Theorem (Vitagliano, W-):
Let J be a holomorphic Jacobi structure on (X, L). The holomorphic line bundle structure L induces a complex structure on $\widehat{j} : D\widehat{L} \to \widehat{D}L$. Moreover, $\widehat{\mathbb{J}} := \begin{pmatrix} \widehat{j} & J^\sharp_{\mathrm{Re}} \\ 0 & -\widehat{j}^* \end{pmatrix}$ is a generalized contact structure on $\widehat{M}$.

A basic example

Example

Let $X = \mathbb{C}^{2n+1}$ together with its standard complex coordinates:

$$\mathbf{p} = \mathbf{p}_1 + i\ \mathbf{p}_2 \quad \mathbf{q} = \mathbf{q}_1 + i\ \mathbf{q}_2 \quad z = x + i\ y$$

Jet J be the holomorphic Jacobi tensor determined by the holomorphic contact structure: $\theta = dz - qdp$. The $\widehat{M} \simeq \mathbb{R}^{4n+2} \times S^1$. The real part of J corresponds to the contact structure $\widehat{\xi} = \mathrm{Ker}\ \widehat{\theta}$ on $\widehat{M}$, where

$$\widehat{\theta} = \cos t\ \theta_x - \sin t\ \theta_y$$

with $\theta_x = dx - \mathbf{q}_1 d\mathbf{p}_1 + \mathbf{q}_2 d\mathbf{p}_2 \quad \text{and} \quad \theta_y = dy - \mathbf{q}_1 d\mathbf{p}_2 - \mathbf{q}_2 d\mathbf{p}_1$

Integration of Holomorphic Jacobi manifold

Definition: A holomorphic Jacobi manifold (X, L, J) is *integrable* if its associated holomorphic Lie algebroid $(\mathcal{J}^1 L)_J$ is integrable.

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We obtained the result:

Theorem (Vitagliano, W-):
A holomorphic Jacobi manifold (X, L, J) is integrable if and only if $(\widehat{M}, \widehat{L}, \widehat{J})$ is integrable. In this case, the global counterpart of $(\mathcal{J}^1 L)_J$ is a contact complex groupoid $(\mathcal{G}, j_{\mathcal{G}}, \xi)$ called the integrating groupoid of (X, L, J).
Moreover, there is a one-to-one correspondence between source-simply connected contact groupoids and integrable holomorphic Jacobi manifolds.

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Remark:

(1) Complex contact manifolds were considered by Kobayashi back in 1959. He noticed that any complex contact manifold (X, ξ) gives rise to principal $U(1)$-bundle over the underlying smooth manifold M. Moreover, M is endowed with a canonical (real) contact structure.

(2) Recently, Crainic and Salazar used their theory of Spencer operators and established the one-to-one correspondence between (real) integrable Jacobi manifolds and source-simply connected contact groupoids. In our proof of the previous theorem we used a different method which uses the homogenization scheme.

THANK YOU !